Chapter 5

Generalized Functions and Green’s Functions

Boundary value problems, involving both ordinary and partial differential equations, can be profitably viewed as the infinite-dimensional function space versions of finite dimensional systems of linear algebraic equations. As a result, linear algebra not only provides us with important insights into their underlying mathematical structure, but also motivates both analytical and numerical solution techniques. In the present chapter, we develop the method of Green’s functions, pioneered by the early nineteenth century self-taught English mathematician (and full-time miller) George Green, whose famous Theorem you already learned in multi-variable calculus. We begin with the simpler case of ordinary differential equations, and then move on to solving the two-dimensional Poisson equation, where the Green’s function provides an powerful alternative to the method of separation of variables.

For inhomogeneous linear systems, the basic Superposition Principle says that the response to a combination of external forces is the self-same combination of responses to the individual forces. In a finite-dimensional system, any forcing function can be decomposed into a linear combination of unit impulse forces, each applied to a single component of the system, and so the full solution can be written as a linear combination of the solutions to the impulse problems. This simple idea will be adapted to boundary value problems governed by differential equations, where the response of the system to a concentrated impulse force is known as the Green’s function. With the Green’s function in hand, the solution to the inhomogeneous system with a general forcing function can be reconstructed by superimposing the effects of suitably scaled impulses. Understanding this construction will become increasingly important as we progress on to partial differential equations, where direct analytical solution techniques are far harder to come by.

The key obstacle preventing the immediate implementation of this idea is that there is no ordinary function that represents an idealized concentrated impulse! Indeed, while this approach was pioneered by Green in the early 1800’s, and then developed into an effective computational tool by the English engineer Oliver Heaviside in the early 1900’s, it took another 50 years before mathematicians were able to develop a completely rigorous theory of generalized functions, also known as distributions. In the language of generalized functions, a unit impulse is represented by a delta function†. While we do not have the analytical tools to completely develop the mathematical theory of generalized functions in its full, rigorous glory, we will spend the first section of the chapter learning the basic concepts.

† Warning: We follow common practice and refer to the “delta distribution” as a function, even though, as we will see, it is most definitely not a function in the usual sense.
and developing the practical computational skills required for applications. The second section will discuss the method of Green’s functions in the context of one-dimensional boundary value problem governed by ordinary differential equations. In the final section, we develop the Green’s function method for solving basic boundary value problems for the two-dimensional Poisson equation.

5.1. Generalized Functions.

To motivate the introduction of objects to represent concentrated impulses, we first review the relevant constructions in the case of linear systems of algebraic equations. Consider a system of \( n \) linear equations in \( n \) unknowns
\[
\mathbf{u} = (u_1, u_2, \ldots, u_n)^T,
\]
written in matrix form
\[
A \mathbf{u} = \mathbf{f},
\] (5.1)
Here \( A \) is a given \( n \times n \) matrix, assumed to be non-singular, \( \det A \neq 0 \), which ensures the existence of a unique solution \( \mathbf{u} \) for any choice of right hand side \( \mathbf{f} = (f_1, f_2, \ldots, f_n)^T \in \mathbb{R}^n \). We regard the linear system (5.1) as representing the equilibrium equations of some physical system, e.g., a system of masses interconnected by springs, [104; Chapter 6]. In this context, the right hand side \( \mathbf{f} \) represents an external forcing, so that its \( i \)th entry, \( f_i \), represents the amount of force exerted on the \( i \)th mass, while the \( i \)th entry of the solution vector, \( u_i \), represents its induced displacement.

Let
\[
\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},
\] (5.2)
denote the standard basis vectors of \( \mathbb{R}^n \). We interpret each \( \mathbf{e}_i \) as a concentrated unit impulse force that is applied solely to the \( i \)th mass in our physical system. Let \( \mathbf{u}_i = (u_{i,1}, \ldots, u_{i,n})^T \) be the induced response of the system, that is, the solution to
\[
A \mathbf{u}_i = \mathbf{e}_i, \quad \text{for each} \quad i = 1, \ldots, n.
\] (5.3)
Let us suppose that we have calculated the response vector \( \mathbf{u}_i \) to each such impulse force. We can express any other force vector as a linear combination,
\[
\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + \cdots + f_n \mathbf{e}_n,
\] (5.4)
\[\text{All vectors are column vectors, but we sometimes use the transpose notation } \mathbf{u}^T, \text{ which indicates the corresponding row vector, in order to save space.}\]
of the impulse forces. The Superposition Principle of Theorem 1.7 then implies that the solution to the inhomogeneous system (5.1) is the self-same linear combination of the individual impulse responses:

\[ u = f_1 u_1 + f_2 u_2 + \cdots + f_n u_n. \]  

(5.5)

Thus, knowing the response of the linear system to the individual impulse forces allows us to immediately determine the response to a general external force.

**Remark:** The alert reader will recognize that \( u_1, \ldots, u_n \) are the columns of the inverse matrix, \( A^{-1} \), and so formula (5.5) is, in fact, reconstructing the solution to the linear system (5.1) by inverting its coefficient matrix: \( u = A^{-1}f \). Thus, this observation does not lead to a genuinely new solution technique for finite-dimensional linear systems.

### The Delta Function

Our aim is to extend this algebraic solution technique to boundary value problems. In this section, we will consider a boundary value problem prescribed by an ordinary differential equation on an interval \( a \leq x \leq b \), the boundary conditions being imposed at the endpoints. We’ve already encountered several examples of boundary value problems in the preceding chapter, and more can be found below.

The key issue is how to characterize an impulse force that is concentrated at a single point. In general, a *unit impulse* at position \( a < \xi < b \) will be described by something called the *delta function*, and denoted by \( \delta_\xi(x) \). Since the impulse is supposed to be concentrated solely at \( x = \xi \), our first requirement is

\[ \delta_\xi(x) = 0 \quad \text{for} \quad x \neq \xi. \]  

(5.6)

Moreover, since the delta function represents a *unit* impulse, we want the total amount of force to be equal to one. Since we are dealing with a continuum, the total force is represented by an integral over the entire interval, and so we also require that the delta function satisfy

\[ \int_a^b \delta_\xi(x) \, dx = 1, \quad \text{provided that} \quad a < \xi < b. \]  

(5.7)

Alas, there is no *bona fide* function that enjoys both of the required properties! Indeed, according to the basic facts of Riemann (or even Lebesgue) integration, two functions which are the same everywhere except at one single point have exactly the same integral, \([38, 111]\). Thus, since \( \delta_\xi \) is zero except at one point, its integral should be 0, not 1. The mathematical conclusion is that the two requirements, (5.6–7) are inconsistent!

This unfortunate fact stopped mathematicians dead in their tracks. It took the imagination of a British engineer, Oliver Heaviside, who was not deterred by the lack of rigorous justification, to start utilizing delta functions in practical applications — with remarkable effect. Despite his success, Heaviside was ridiculed by the pure mathematicians of his day, and eventually succumbed to mental illness. But, some thirty years later, the great theoretical physicist Paul Dirac resurrected the delta function for quantum mechanical applications, and this finally made the mathematicians sit up and take notice. (Indeed,
the term “Dirac delta function” is quite common.) In 1944, the French mathematician Laurent Schwartz finally established a rigorous theory of distributions that incorporated such useful, but non-standard objects, \[77, 110\]. Thus, to be more accurate, we should really refer to the delta distribution; however, we will retain the more common, intuitive designation “delta function” throughout. It is beyond the scope of this introductory text to develop a fully rigorous theory of distributions. Rather, in the spirit of Heaviside, we shall concentrate on learning, through practice with computations and applications, how to make effective use of these exotic mathematical creatures.

There are two possible ways to introduce the delta function. Both are important and both worth understanding.

Method #1. Limits: The first approach is to regard the delta function \(\delta_\xi(x)\) as a limit of a sequence of ordinary smooth functions\(^\ddagger\) \(g_n(x)\). These will represent progressively more and more concentrated unit forces, which, in the limit, converge to the desired unit impulse concentrated at a single point, \(x = \xi\). Thus, we require

\[
\lim_{n \to \infty} g_n(x) = 0, \quad x \neq \xi, \tag{5.8}
\]

while the total amount of force remains fixed at

\[
\int_a^b g_n(x) \, dx = 1. \tag{5.9}
\]

On a formal level, the limit “function”

\[
\delta_\xi(x) = \lim_{n \to \infty} g_n(x)
\]

will satisfy the key properties (5.6–7).

An explicit example of such a sequence is provided by the rational functions

\[
g_n(x) = \frac{n}{\pi(1 + n^2 x^2)}. \tag{5.10}
\]

These functions satisfy

\[
\lim_{n \to \infty} g_n(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \tag{5.11}
\]

while\(^\ddagger\)

\[
\int_{-\infty}^{\infty} g_n(x) \, dx = \frac{1}{\pi} \tan^{-1} n x \bigg|_{x=-\infty}^{x=\infty} = 1. \tag{5.12}
\]

Therefore, formally, we identify the limiting function

\[
\lim_{n \to \infty} g_n(x) = \delta(x) = \delta_0(x), \tag{5.13}
\]

\(^\ddagger\) To keep the notation compact, we suppress the dependence of the functions \(g_n\) on the point \(\xi\) where the limiting delta function is concentrated.

\(^\ddagger\) For the moment, it will be slightly simpler to consider the entire real line. Exercise \(\blacksquare\) discusses how to adapt the construction to a finite interval.
with the unit impulse delta function concentrated at $x = 0$. As sketched in Figure 5.1, as $n$ gets larger and larger, each successive function $g_n(x)$ forms a more and more concentrated spike, while maintaining a unit total area under its graph. The limiting delta function can be thought of as an infinitely tall spike of zero width, entirely concentrated at the origin.

Remark: There are many other possible choices for the limiting functions $g_n(x)$. See Exercise $\blacksquare$ for another important example.

Remark: This construction of the delta function highlights the perils of interchanging limits and integrals without rigorous justification. In any standard theory of integration (Riemann integrals, Lebesgue integrals, etc.), the limit of the functions $g_n$ would be indistinguishable from the zero function, so the limit of their integrals (5.12) would not equal the integral of their limit:

$$1 = \lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(x) \, dx \neq \int_{-\infty}^{\infty} \lim_{n \to \infty} g_n(x) \, dx = 0.$$ 

The delta function is, in a sense, a means of sidestepping this analytic inconvenience. The full ramifications and theoretical constructions underlying such limits must, however, be deferred to a rigorous course in real analysis, [38, 111].

Once we have defined the basic delta function $\delta(x) = \delta_0(x)$ concentrated at the origin, we can obtain the delta function concentrated at any other position $\xi$ by a simple translation:

$$\delta_\xi(x) = \delta(x - \xi). \quad (5.14)$$
Thus, \( \delta_\xi(x) \) can be realized as the limit, as \( n \to \infty \), of the translated functions

\[
\hat{g}_n(x) = g_n(x - \xi) = \frac{n}{\pi \left[ 1 + n^2(x - \xi)^2 \right]}.
\]

\( (5.15) \)

**Method #2. Duality:** The second approach is a bit more abstract, but much closer to the proper rigorous formulation of the theory of distributions like the delta function. The critical property is that if \( u(x) \) is any continuous function, then

\[
\int_a^b \delta_\xi(x) u(x) \, dx = u(\xi), \quad \text{for} \quad a < \xi < b.
\]

\( (5.16) \)

Indeed, since \( \delta_\xi(x) = 0 \) for \( x \neq \xi \), the integrand only depends on the value of \( u \) at the point \( x = \xi \), and so

\[
\int_a^b \delta_\xi(x) u(x) \, dx = \int_a^b \delta_\xi(x) u(\xi) \, dx = u(\xi) \int_a^b \delta_\xi(x) \, dx = u(\xi).
\]

Equation (5.16) serves to define a linear functional† \( L_\xi : C^0[a, b] \to \mathbb{R} \) that maps a continuous function \( u \in C^0[a, b] \) to its value at the point \( x = \xi \):

\[
L_\xi[u] = u(\xi).
\]

\( (5.17) \)

The basic linearity requirements (1.11) are immediately established:

\[
L_\xi[u + v] = u(\xi) + v(\xi) = L_\xi[u] + L_\xi[v], \quad L_\xi[c u] = c u(\xi) = c L_\xi[u],
\]

for any functions \( u(x), v(x) \). In the dual approach to generalized functions, the delta function is, in fact, defined as this particular linear functional (5.17). The function \( u(x) \) is sometimes referred to as a test function, since it serves to “test” the form of the linear functional \( L_\xi \).

**Remark:** If the impulse point \( \xi \) lies outside the integration domain, then

\[
\int_a^b \delta_\xi(x) u(x) \, dx = 0, \quad \text{whenever} \quad \xi < a \quad \text{or} \quad \xi > b,
\]

because the integrand is identically zero on the entire interval. For technical reasons, we will not attempt to define the integral (5.18) if the impulse point \( \xi = a \) or \( \xi = b \) lies on the boundary of the interval of integration.

The interpretation of the linear functional \( L_\xi \) as representing a kind of function \( \delta_\xi(x) \) is based on the following line of thought. According to Corollary B.33, every scalar-valued linear function \( L : \mathbb{R}^n \to \mathbb{R} \) on the finite-dimensional vector space \( \mathbb{R}^n \) is given by taking the dot product with a fixed element \( a \in \mathbb{R}^n \), so

\[
L[u] = a \cdot u.
\]

† The term “functional” serves to indicate a linear function whose domain is a function space, thus avoiding confusion with the functions in its domain.
In this sense, linear functions on $\mathbb{R}^n$ are the “same” as vectors. Similarly, on the infinite-dimensional function space $C^0[a,b]$, the $L^2$ inner product taken with a fixed continuous function $g \in C^0[a,b]$,

$$L_g[u] = \langle g ; u \rangle = \int_a^b g(x) u(x) \, dx,$$

(5.19)
defines a real-valued linear functional $L_g : C^0[a,b] \rightarrow \mathbb{R}$. However, unlike the finite-dimensional situation, not every real-valued linear functional has this form! In particular, there is no actual function $\delta_{\xi}(x)$ such that the identity

$$L_{\delta_{\xi}}[u] = \langle \delta_{\xi} ; u \rangle = \int_a^b \delta_{\xi}(x) u(x) \, dx = u(\xi)$$

(5.20)
holds for every continuous function $u(x)$. Every (continuous) function defines a linear functional, but not conversely.

But the dual interpretation of generalized functions acts as if this were true. Generalized functions are, in actuality, real-valued linear functionals on function space. However, they are viewed as a kind of function via the $L^2$ inner product. Although this identification is not to be taken too literally, one can, with some care, manipulate generalized functions as if they were actual functions, but always keeping in mind that a rigorous justification of such computations must ultimately rely on their innate characterization as linear functionals.

The two approaches — limits and duality — are completely compatible. Indeed, one can recover the dual formula (5.20) as the limit

$$u(\xi) = \lim_{n \to \infty} \langle g_n ; u \rangle = \lim_{n \to \infty} \int_a^b g_n(x) u(x) \, dx = \int_a^b \delta_{\xi}(x) u(x) \, dx = \langle \delta_{\xi} ; u \rangle$$

(5.21)
of the inner products of the function $u$ with the approximating concentrated impulse functions $g_n(x)$ satisfying (5.8–9). In this manner, the limiting linear functional represents the delta function:

$$u(\xi) = L_{\delta_{\xi}}[u] = \lim_{n \to \infty} L_n[u], \quad \text{where} \quad L_n[u] = \int_0^\ell g_n(x) u(x) \, dx.$$

Thus, the choice of interpretation of the generalized delta function is, on an operational level, a matter of taste. For the novice, the limit version is perhaps easier to digest at first. However, the dual, linear functional interpretation has stronger connections with the rigorous theory and, even in applications, offers some significant advantages.

Although to the novice, the delta function might seem a little bizarre, its utility in modern applied mathematics and mathematical physics more than justifies including it in your analytical toolbox. Even though you are probably not yet comfortable with either definition, you are advised to press on and familiarize yourself with its basic properties. With a little care, you usually won’t go far wrong by treating it as if it were a genuine function. After you gain more practical experience, you can, if desired, return to contemplate just exactly what kind of creature the delta function really is.
Calculus of Generalized Functions

In order to make use of the delta function, we need to understand how it behaves under the basic operations of linear algebra and calculus. First, we can take linear combinations of delta functions. For example,

\[ h(x) = 2\delta(x) - 3\delta(x-1) = 2\delta_0(x) - 3\delta_1(x) \]

represents a combination of an impulse of magnitude 2 concentrated at \( x = 0 \) and one of magnitude \(-3\) concentrated at \( x = 1 \). In the dual interpretation, \( f \) defines the linear functional

\[ L_h[u] = \langle h; u \rangle = 2\langle \delta_0; u \rangle - 3\langle \delta_1; u \rangle = 2u(0) - 3u(1), \]

or, more explicitly,

\[ L_h[u] = \int_a^b h(x) u(x) \, dx = \int_a^b [2\delta(x) - 3\delta(x-1)] u(x) \, dx = 2 \int_a^b \delta(x) u(x) \, dx - 3 \int_a^b \delta(x-1) u(x) \, dx = 2u(0) - 3u(1), \]

provided \( a < 0 \) and \( b > 1 \).

Next, since \( \delta_\xi(x) = 0 \) for any \( x \neq \xi \), multiplying the delta function by an ordinary function is the same as multiplying by a constant:

\[ g(x) \delta_\xi(x) = g(\xi) \delta_\xi(x), \quad (5.22) \]

provided that \( g(x) \) is continuous at \( x = \xi \). For example, \( x\delta(x) \equiv 0 \) is the same as the constant zero function.

Warning: Since they are inherently linear functionals, it is not permissible to multiply delta functions together, or to apply more complicated algebraic operations to them. Expressions like \( \delta(x)^2 \), \( 1/\delta(x) \), \( e^{\delta(x)} \), etc., are not well defined in the theory of generalized functions. This makes their application to nonlinear differential equations considerably more problematic.

Figure 5.2.  Step Function as Limit.
The integral of the delta function is the unit step function:
\[
\int_a^x \delta_\xi(t) \, dt = \sigma_\xi(x) = \sigma(x - \xi) = \begin{cases} 
0, & x < \xi, \\
1, & x > \xi,
\end{cases} \quad \text{provided } a < \xi. \tag{5.23}
\]
Unlike the delta function, the step function \( \sigma_\xi(x) \) is an ordinary function. It is continuous — indeed constant — except at \( x = \xi \). The value of the step function at the discontinuity \( x = y \) is left unspecified, although a wise choice — compatible with Fourier theory — is to set \( \sigma_\xi(y) = \frac{1}{2} \), the average of its left and right hand limits.

We note that the integration formula (5.23) is compatible with our characterization of the delta function as the limit of highly concentrated forces. Integrating the approximating functions (5.10), we obtain
\[
f_n(x) = \int_{-\infty}^x g_n(t) \, dt = \frac{1}{\pi} \tan^{-1} n x + \frac{1}{2}.
\]
Since
\[
\lim_{y \to \infty} \tan^{-1} y = \frac{1}{2} \pi, \quad \text{while} \quad \lim_{y \to -\infty} \tan^{-1} y = -\frac{1}{2} \pi,
\]
these functions converge to the step function:
\[
\lim_{n \to \infty} f_n(x) = \sigma(x) = \begin{cases} 
1, & x > 0, \\
\frac{1}{2}, & x = 0, \\
0, & x < 0.
\end{cases} \tag{5.24}
\]
A graphical illustration of this limiting process appears in Figure 5.2.

The integral of the discontinuous step function (5.23) is the continuous ramp function
\[
\int_a^x \sigma_\xi(t) \, dt = \rho_\xi(x) = \rho(x - \xi) = \begin{cases} 
0, & x < \xi, \\
x - \xi, & x > \xi,
\end{cases} \quad \text{provided } a < \xi, \tag{5.25}
\]
which is graphed in Figure 5.3. Note that \( \rho(x - \xi) \) has a corner at \( x = \xi \), and so is not differentiable there; indeed, its derivative \( \frac{d\rho}{dx} = \sigma \) has a jump discontinuity. We can continue to integrate; the \( n \)th integral of the delta function is the \( n \)th order ramp function
\[
\rho_n(x - \xi) = \begin{cases} 
\frac{(x - \xi)^n}{n!}, & x > \xi, \\
0, & x < \xi.
\end{cases} \tag{5.26}
\]
What about differentiation? Motivated by the Fundamental Theorem of Calculus, we shall use formula (5.23) to identify the derivative of the step function with the delta function
\[
\frac{d\sigma}{dx} = \delta.
\] (5.27)
This fact is highly significant. In basic calculus, one is not allowed to differentiate a discontinuous function. Here, we discover that the derivative can be defined, not as an ordinary function, but rather as a generalized delta function.

This basic identity is a particular instance of a general rule for differentiating functions with discontinuities. The function \( f(x) \) is continuous at the point \( \xi \) if and only if its left and right sided limits
\[
f(\xi^-) = \lim_{x \to \xi^-} f(x), \quad f(\xi^+) = \lim_{x \to \xi^+} f(x),
\] (5.28)
exist and are equal to its value at the point:
\[
f(\xi) = f(\xi^-) = f(\xi^+).\] (5.29)
If the one-sided limits are the same, but not equal to \( f(\xi) \), then the function is said to have a removable discontinuity, since redefining its value according to (5.29) makes \( f \) continuous at the point in question. An example is the function \( f(x) \) that is equal to 0 for all \( x \neq 0 \), but has\(^\dagger\) \( f(0) = 1 \). Removing the discontinuity by setting \( f(0) = 0 \) makes \( f(x) \equiv 0 \) equal to a continuous constant function. Since removable discontinuities have no effect in either the theory or applications, they can always be removed without penalty.

**Warning**: Although \( \delta(0^+) = 0 = \delta(0^-) \), we will emphatically not call 0 a removable discontinuity of the delta function. Only standard functions have removable discontinuities.

Finally, if both the left and right limits exist, but are not equal, then \( f \) is said to have a jump discontinuity at the point \( \xi \). The magnitude of the jump is the difference
\[
\beta = f(\xi^+) - f(\xi^-) = \lim_{x \to \xi^+} f(x) - \lim_{x \to \xi^-} f(x)
\] (5.30)
between the right and left limits. The magnitude of the jump is positive if the function jumps up, when moving from left to right, and negative if it jumps down. Note the value of the function at the point, namely \( f(\xi) \) — which may not even be defined — plays no role in the specification of the jump. For example, the step function \( \sigma(x) \) has a unit, i.e., magnitude 1, jump discontinuity at the origin:
\[
\sigma(0^+) - \sigma(0^-) = 1 - 0 = 1,
\]
and is continuous everywhere else.

In general, the derivative of a function with jump discontinuities is a generalized function that includes delta functions concentrated at each discontinuity. More explicitly,

\(^\dagger\) This function is not a version of the delta function. It is an ordinary function, and its integral is 0, not 1.
suppose that \( f(x) \) is differentiable, in the usual calculus sense, everywhere except at a point \( y \) where it has a jump discontinuity of magnitude \( \beta \). We can re-express the function in the convenient form

\[
 f(x) = g(x) + \beta \sigma(x - \xi), \tag{5.31}
\]

where \( g(x) \) is continuous everywhere, with a removable discontinuity at \( x = \xi \), and differentiable except possibly at the jump. Differentiating (5.31), we find that

\[
 f'(x) = g'(x) + \beta \delta(x - \xi), \tag{5.32}
\]

has a delta spike of magnitude \( \beta \) at the discontinuity. Thus, the derivatives of \( f \) and \( g \) coincide everywhere except at the discontinuity.

**Example 5.1.** Consider the function

\[
 f(x) = \begin{cases} 
 -x, & x < 1, \\
 \frac{1}{5}x^2, & x > 1,
\end{cases} \tag{5.33}
\]

which we graph in Figure 5.4. We note that \( f \) has a single jump discontinuity at \( x = 1 \) of magnitude

\[
 f(1^+) - f(1^-) = \frac{1}{5} - (-1) = \frac{6}{5}.
\]

This means that

\[
 f(x) = g(x) + \frac{6}{5} \sigma(x - 1), \quad \text{where} \quad g(x) = \begin{cases} 
 -x, & x < 1, \\
 \frac{1}{5}x^2 - \frac{6}{5}, & x > 1,
\end{cases}
\]

is continuous everywhere, since its right and left hand limits at the original discontinuity are equal: \( g(1^+) = g(1^-) = -1 \). Therefore,

\[
 f'(x) = g'(x) + \frac{6}{5} \delta(x - 1), \quad \text{where} \quad g'(x) = \begin{cases} 
 -1, & x < 1, \\
 \frac{2}{5}x, & x > 1,
\end{cases}
\]

while \( g'(1) \) and \( f'(1) \) are not defined. In Figure 5.4, the delta spike in the derivative of \( f \) is symbolized by a vertical line — although this pictorial device fails to indicate its magnitude of \( \frac{6}{5} \).
Note that, in this particular example, \( g'(x) \) can be found by directly differentiating the formula for \( f(x) \). Indeed, in general, once we determine the magnitude and location of the jump discontinuities of \( f(x) \), we can compute its derivative directly without introducing to the auxiliary function \( g(x) \).

**Example 5.2.** As a second, more streamlined example, consider the function
\[
f(x) = \begin{cases} 
-x, & x < 0, \\
x^2 - 1, & 0 < x < 1, \\
2e^{-x}, & x > 1,
\end{cases}
\]
which is plotted in Figure 5.5. This function has jump discontinuities of magnitude \(-1\) at \( x = 0 \), and of magnitude \(2/e\) at \( x = 1 \). Therefore, in light of the preceding remark,
\[
f'(x) = -\delta(x) + \frac{2}{e} \delta(x - 1) + \begin{cases} 
-1, & x < 0, \\
2x, & 0 < x < 1, \\
-2e^{-x}, & x > 1,
\end{cases}
\]
where the final terms are obtained by directly differentiating \( f(x) \).

**Example 5.3.** The derivative of the absolute value function
\[
a(x) = |x| = \begin{cases} 
x, & x > 0, \\
-x, & x < 0,
\end{cases}
\]
is the *sign function*
\[
a'(x) = \text{sign } x = \begin{cases} 
+1, & x > 0, \\
-1, & x < 0.
\end{cases}
\]
(5.34)
Note that there is no delta function in \( a'(x) \) because \( a(x) \) is continuous everywhere. Since \( \text{sign } x \) has a jump of magnitude 2 at the origin and is otherwise constant, its derivative is twice the delta function:
\[
a''(x) = \frac{d}{dx} \text{sign } x = 2\delta(x).
\]
Example 5.4. We are even allowed to differentiate the delta function. Its first derivative
\[ \delta_\xi'(x) = \delta'(x - \xi) \] (5.35)
can be interpreted in two ways. First, we may view \( \delta'(x) \) as the limit of the derivatives of the approximating functions (5.10):

\[ \frac{d\delta}{dx} = \lim_{n \to \infty} \frac{dg_n}{dx} = \lim_{n \to \infty} \frac{-2n^3x}{\pi(1 + n^2x^2)^2}. \] (5.36)

The graphs of these rational functions take the form of more and more concentrated spiked “doublets”, as illustrated in Figure 5.6. To determine the effect of the derivative on a test function \( u(x) \), we compute the limiting integral

\[ \langle \delta'; u \rangle = \int_{-\infty}^{\infty} \delta'(x)u(x)\,dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} g'_n(x)u(x)\,dx \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(x)\,u'(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\,u'(x)\,dx = -u'(0). \] (5.37)

In the middle step, we used an integration by parts, noting that the boundary terms at \( \pm \infty \) vanish, provided that \( u(x) \) is continuously differentiable and bounded as \( |x| \to \infty \). Pay attention to the minus sign in the final answer.

In the dual interpretation, the generalized function \( \delta_\xi'(x) \) corresponds to the linear functional

\[ L'_\xi[u] = -u'(\xi) = \langle \delta_\xi'; u \rangle = \int_{a}^{b} \delta_\xi'(x)\,u(x)\,dx, \quad \text{where} \quad a < \xi < b, \] (5.38)
that maps a continuously differentiable function $u(x)$ to minus its derivative at the point $y$. We note that (5.38) is compatible with a formal integration by parts:

$$\int_a^b \delta'(x - \xi) u(x) \, dx = \left. \delta(x - \xi) u(x) \right|_{x=a}^{x=b} - \int_a^b \delta(x - \xi) u'(x) \, dx = -u'(\xi).$$

The boundary terms at $x = a$ and $x = b$ automatically vanish since $\delta(x - \xi) = 0$ for $x \neq \xi$.

**Warning:** The functions $\tilde{g}_n(x) = g_n(x) + g'_n(x)$ satisfy $\lim_{n \to \infty} \tilde{g}_n(x) = 0$ for all $x \neq y$, while $\int_{-\infty}^{\infty} \tilde{g}_n(x) \, dx = 1$. However, $\lim_{n \to \infty} \tilde{g}_n = \lim_{n \to \infty} g_n + \lim_{n \to \infty} g'_n = \delta + \delta'$. Thus, our original conditions (5.8–9) are not in fact sufficient to characterize whether a sequence of functions has the delta function as a limit. To be absolutely sure, one must, in fact, verify the more comprehensive limiting formula (5.21).

**The Fourier Series of the Delta Function**

Let us next investigate extending the method of Fourier series to represent generalized functions. Let us begin with the basic delta function $\delta(x)$, based at the origin. Using the characterizing properties (5.16), its real Fourier coefficients are

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x) \cos k x \, dx = \frac{1}{\pi} \cos k 0 = \frac{1}{\pi},$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x) \sin k x \, dx = \frac{1}{\pi} \sin k 0 = 0.$$

Therefore, the Fourier series is

$$\delta(x) \sim \frac{1}{2\pi} + \frac{1}{\pi} \left( \cos x + \cos 2x + \cos 3x + \cdots \right).$$

(5.40)

Since $\delta(x) = \delta(-x)$ is an even function (Why?), it should come as no surprise that it has a cosine series. Alternatively, we can rewrite the series in complex form

$$\delta(x) \sim \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} = \frac{1}{2\pi} \left( \cdots + e^{-2ix} + e^{-ix} + 1 + e^{ix} + e^{2ix} + \cdots \right).$$

(5.41)

where the complex Fourier coefficients are computed† as

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x) e^{-ikx} \, dx = \frac{1}{2\pi}.$$

† While we can test the delta function with any continuous function, we are only permitted to test its derivative on continuously differentiable functions. To avoid such technicalities, one often only restricts to completely smooth, meaning infinitely differentiable, test functions.

† Or, we could use (3.66).
Remark: Although we stated that the Fourier series (5.40) represents the delta function, this is not entirely correct. Remember that a Fourier series converges to the $2\pi$ periodic extension of the original function. Therefore, (5.41) actually represents the periodic extension of the delta function:

$$\tilde{\delta}(x) = \cdots + \delta(x + 4\pi) + \delta(x + 2\pi) + \delta(x) + \delta(x - 2\pi) + \delta(x - 4\pi) + \delta(x - 6\pi) + \cdots,$$

(5.42)
consisting of a periodic array of unit impulses concentrated at all integer multiples of $2\pi$.

Let us investigate in what sense (if any) the Fourier series (5.40) or, equivalently, (5.41), represents the delta function. The first observation is that, because its summands do not tend to zero, the series certainly doesn’t converge in the usual, calculus sense. Nevertheless, in a very generalized sense, the series can be regarded as converging to the (periodic extension of the) delta function.

To understand the convergence mechanism, we recall that we already established a formula (3.125) for the partial sums:

$$s_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{n} \cos kx = \frac{1}{2\pi} \sin \left(\frac{n+\frac{1}{2}}{2}\right) x.$$ 

(5.43)

Graphs of some of the partial sums on the interval $[-\pi, \pi]$ are displayed in Figure 5.7. Note that, as $n$ increases, the spike at $x = 0$ becomes progressively becomes taller and thinner, converging to an infinitely tall delta spike. (We have had to truncate the last two graphs; the spike extends beyond the top.) Indeed, by l’Hôpital’s Rule,

$$\lim_{x \to 0} \frac{1}{2\pi} \frac{\sin \left(\frac{n+\frac{1}{2}}{2}\right) x}{\sin \frac{1}{2} x} = \lim_{x \to 0} \frac{1}{2\pi} \frac{\left(n+\frac{1}{2}\right) \cos \left(n+\frac{1}{2}\right) x}{\frac{1}{2} \cos \frac{1}{2} x} \frac{n+\frac{1}{2}}{\pi} \to \infty \text{ as } n \to \infty.$$
(An elementary proof of this formula is to note that, at \( x = 0 \), every term in the original sum (5.40) is equal to 1.) Furthermore, the integrals remain fixed,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} s_n(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \left( n + \frac{1}{2} \right) x \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-n}^{n} e^{ikx} \, dx = 1, \tag{5.44}
\]
as required for convergence to the delta function. However, away from the spike, the partial sums do not go to zero! Rather, they oscillate ever more rapidly, while maintaining a fixed overall amplitude of

\[
\frac{1}{2\pi} \csc \frac{1}{2} x = \frac{1}{2\pi \sin \frac{1}{2} x}. \tag{5.45}
\]

As \( n \) gets large, the amplitude function (5.45) can be seen as the envelope of the increasingly rapid oscillations. So, roughly speaking, the convergence \( s_n(x) \to \delta(x) \) means that the “infinitely fast” oscillations are somehow canceling each other out, and the net effect is zero away from the spike at \( x = 0 \). The convergence of the Fourier sums to \( \delta(x) \) is much more subtle than in the original limiting definition (5.10).

The technical term is weak convergence, which plays a very important role in advanced mathematical analysis, [112], signal processing, composite materials, and elsewhere.

**Definition 5.5.** A sequence of functions \( f_n(x) \) is said to converge weakly to \( f_\ast(x) \) on an interval \([a, b]\) if the \( L^2 \) inner products

\[
\int_a^b f_n(x) u(x) \, dx \longrightarrow \int_a^b f_\ast(x) u(x) \, dx \quad \text{as} \quad n \longrightarrow \infty, \tag{5.46}
\]

for every continuous test function \( u(x) \in C^0[a, b] \). Weak convergence is often indicated by a half-pointed arrow: \( f_n \rightharpoonup g \).

**Remark:** On unbounded intervals, one usually restricts the test functions to have compact support meaning that \( u(x) = 0 \) for all sufficiently large \(|x| \gg 0\). One can also restrict to smooth test functions only, e.g., require that \( u \in C^\infty[a, b] \). Clearly, weak convergence with continuous test functions implies weak convergence with smooth test functions, but the converse may not be valid. Indeed, the fewer test functions available, the more chance one has that the weak convergence criterion (5.46) holds.

**Example 5.6.** Let us show that the trigonometric functions \( f_n(x) = \cos nx \) converge weakly to the zero function:

\[
\cos nx \to 0 \quad \text{as} \quad n \to \infty \quad \text{on the interval } [-\pi, \pi].
\]

(Actually, this holds true on any interval; see Exercise [ ].) According to the definition, we need to prove that

\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} u(x) \cos nx \, dx = 0,
\]
for any continuous function \( u \in C^0[-\pi, \pi] \). But this is just a restatement of the Riemann–
Lebesgue Lemma 3.35, which states that the high frequency Fourier coefficients of a continuous
(indeed, even square integrable) function \( u(x) \) go to zero. The same remark establishes
the weak convergence of \( \sin n x \rightharpoonup 0 \).

Observe that the functions \( \cos n x \) fail to converge pointwise to 0 at any value of \( x \). Indeed, if \( x \) is an integer multiple of \( 2\pi \), then \( \cos n x = 1 \) for all \( n \). If \( x \) is any other
rational multiple of \( \pi \), the values of \( \cos n x \) periodically cycle through a finite number of
different values, and never go to 0, while if \( x \) is an irrational multiple of \( \pi \), they oscillate
aperiodically between \(-1\) and \(+1\). The functions also fail to converge in norm to 0, since
their (unscaled) \( L^2 \) norms remain fixed at

\[
\| \cos n x \| = \sqrt{\int_{-\pi}^{\pi} \cos^2 n x \, dx} = \sqrt{\pi} \quad \text{for all} \quad n > 0.
\]

The cancellation of such oscillations in the high frequency limit is a characteristic feature
of weak convergence.

Let us now explain why, although the Fourier series (5.40) does not converge to the
delta function either pointwise, or in norm (indeed, \( \| \delta \| \) is not even defined!), it does
converge weakly on \([-\pi, \pi]\). More specifically, we need to prove that the partial sums
\( s_n \rightharpoonup \delta \), meaning that

\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} s_n(x) u(x) \, dx = \int_{-\pi}^{\pi} \delta(x) u(x) \, dx = u(0)
\]

for every continuous function \( u \in C^0[-\pi, \pi] \), or, equivalently,

\[
\lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \frac{\sin \left( n + \frac{1}{2} \right) x}{\sin \frac{1}{2} x} \, dx = u(0).
\]

But this is a restatement of a special case of the identities (3.126) used in the proof
of the Pointwise Convergence Theorem 3.8 for Fourier series. Indeed, summing the two
identities in (3.126), and then setting \( x = 0 \) reproduces (5.48) since, by continuity,
\( u(0) = \frac{1}{2} \left[ u(0^+) + u(0^-) \right] \). In other words, the pointwise convergence of the Fourier series of a
continuous function is equivalent to the weak convergence of the Fourier series of the delta
function!

It is a remarkable, profound fact that Fourier analysis is entirely compatible with the
calculus of generalized functions. For instance, term-wise differentiation of the Fourier
series for a piecewise \( C^1 \) function leads to the Fourier series for the differentiated function
that incorporates delta functions of the appropriate magnitude at each jump discontinuity.
This fact further reassures us that the rather mysterious construction of delta functions
and their generalizations is indeed the right way to extend calculus to functions which do
not possess derivatives in the ordinary sense.

**Example 5.7.** If we differentiate the Fourier series

\[
x = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin kx = 2 \left( \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \right),
\]

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we obtain an apparent contradiction:

\[ 1 \sim 2 \sum_{k=1}^{\infty} (-1)^{k+1} \cos kx = 2 \cos x - 2 \cos 2x + 2 \cos 3x - 2 \cos 4x + \cdots . \tag{5.49} \]

But the Fourier series for 1 just consists of a single constant term! (Why?)

The resolution of this paradox is not difficult. The Fourier series (3.37) does not converge to \( x \), but rather to its periodic extension \( \tilde{f}(x) \), which has a jump discontinuity of magnitude \( 2\pi \) at odd multiples of \( \pi \); see Figure 3.1. Thus, Theorem 3.20 is not directly applicable. Nevertheless, we can assign a consistent interpretation to the differentiated series. The derivative \( \tilde{f}'(x) \) of the periodic extension is not equal to the constant function 1, but, rather, has an additional delta function concentrated at each jump discontinuity:

\[
\tilde{f}'(x) = 1 - 2\pi \sum_{j=-\infty}^{\infty} \delta(x - (2j + 1)\pi) = 1 - 2\pi \tilde{\delta}(x - \pi),
\]

where \( \tilde{\delta} \) denotes the \( 2\pi \) periodic extension of the delta function, cf. (5.42). The differentiated Fourier series (5.49) does, in fact, represent \( \tilde{f}'(x) \). Indeed, the Fourier coefficients of \( \tilde{\delta}(x - \pi) \) are

\[
a_k = \frac{1}{\pi} \int_{0}^{2\pi} \delta(x - \pi) \cos kx \, dx = \frac{1}{\pi} \cos k\pi = \frac{(-1)^k}{\pi},
\]

\[
b_k = \frac{1}{\pi} \int_{0}^{2\pi} \delta(x - \pi) \sin kx \, dx = \frac{1}{\pi} \sin k\pi = 0,
\]

Observe that we changed the interval of integration to \([0, 2\pi]\) to avoid placing the delta function singularities at the endpoints. Thus,

\[
\delta(x - \pi) \sim \frac{1}{2\pi} + \frac{1}{\pi}(-\cos x + \cos 2x - \cos 3x + \cdots), \tag{5.50}
\]

which serves to justify the result.

**Example 5.8.** Let us differentiate the Fourier series

\[
\sigma(x) \sim \frac{1}{2} + \frac{2}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots \right)
\]

for the step function we found in Example 3.9, and see if we end up with the Fourier series (5.40) for the delta function. We find

\[
\frac{d}{dx} \sigma(x) \sim \frac{2}{\pi} \left( \cos x + \cos 3x + \cos 5x + \cos 7x + \cdots \right), \tag{5.51}
\]

which does not agree with (5.40) — half the terms are missing! The explanation is similar to the preceding example: the \( 2\pi \) periodic extension \( \tilde{\sigma}(x) \) of the step function has two jump discontinuities, of magnitudes +1 at even multiples of \( \pi \) and -1 at odd multiples; see Figure 3.6. Therefore, its derivative

\[
\tilde{\sigma}(x) = \tilde{\delta}(x) - \tilde{\delta}(x - \pi)
\]
is the difference of the $2\pi$ periodic extension of the delta function at 0, with Fourier series (5.40), minus the $2\pi$ periodic extension of the delta function at $\pi$, with Fourier series (5.50). The difference of these two delta function series produces (5.51).

5.2. Green’s Functions for One–Dimensional Boundary Value Problems.

In this section, we will put the delta function to work by developing a general method for solving inhomogeneous linear boundary value problems. The key idea, motivated by the linear algebra method outlined at the beginning of the previous section, is to first solve the system subjected to a unit delta function impulse. The resulting solution is known as the Green’s function. We can then use a Linear Superposition Principle to write down the solution for a general forcing inhomogeneity. The method is very general, but will be developed in the context of a few relatively simple examples.

Example 5.9. The boundary value problem
\[- cu'' = f(x), \quad u(0) = 0 = u(1), \tag{5.52}\]
models the deformation $u(x)$ of a homogeneous bar of unit length and constant stiffness $c$ that is fixed at both ends that is subject to an external force $f(x)$. The Green’s function for this particular boundary value problem is the family of solutions
\[ u(x) = G_\xi(x) = G(x; \xi) \]
to the case when the forcing function is a unit impulse concentrated at a point $0 < \xi < 1$:
\[- cu'' = \delta(x - \xi), \quad u(0) = 0 = u(1). \tag{5.53}\]
As usual, to solve such a boundary value problem, we first solve the inhomogeneous differential equation, and then fix the integration constants by substituting into the boundary conditions.

The solution to the differential equation can be obtained by direct integration. First, by (5.23),
\[ u'(x) = - \frac{\sigma(x - \xi)}{c} + a, \]
where $a$ is a constant of integration. A second integration leads to
\[ u(x) = - \frac{\rho(x - \xi)}{c} + ax + b, \tag{5.54} \]
where $\rho$ is the ramp function (5.25). The integration constants $a, b$ are fixed by the boundary conditions; since $0 < \xi < 1$, we have
\[ u(0) = b = 0, \quad \text{while} \quad u(1) = - \frac{1 - \xi}{c} + a + b = 0, \quad \text{and so} \quad a = \frac{1 - \xi}{c}. \]
Therefore, the Green’s function for the problem is
\[ G(x; \xi) = - \rho(x - \xi) + (1 - \xi)x = \begin{cases} x(1 - \xi)/c, & x \leq \xi, \\ (1 - \xi)x/c, & x \geq \xi, \end{cases} \tag{5.55} \]
Figure 5.8. Green’s function for a Bar with Fixed Ends.

Figure 5.8 sketches a graph of $G(x; \xi)$ at one particular value of $y = .7$, in the case $c = 1$. Note that, for each fixed $\xi$, $G_{\xi}(x) = G(x; \xi)$ is a continuous function of $x$; its graph consists of two connected straight line segments, with a corner at the point of application of the unit impulse force.

Once we have determined the Green’s function, we are able to solve the general inhomogeneous boundary value problem (5.52) by superposition. We first express the forcing function $f(x)$ as a linear combination of impulses concentrated at points along the bar. Since there is a continuum of possible positions $0 < \xi < 1$ at which impulse forces may be applied, we will use an integral to sum them up, thereby writing the external force as

$$f(x) = \int_0^1 \delta(x - \xi) f(\xi) \, d\xi. \quad (5.56)$$

We can interpret (5.56) as the (continuous) superposition of an infinite collection of impulses, $f(\xi) \delta(x - \xi)$, of magnitude $f(\xi)$ and concentrated at position $\xi$.

The Superposition Principle states that linear combinations of inhomogeneities produce linear combinations of solutions. Again, we adapt this principle to the continuum by replacing the sums by integrals. Thus, we claim that the solution to the boundary value problem is the self-same linear superposition

$$u(x) = \int_0^1 G(x; \xi) f(\xi) \, d\xi \quad (5.57)$$

of the Green’s function solutions to the individual unit impulse problems.

For the particular boundary value problem (5.52), we use the explicit formula (5.55) for the Green’s function. Breaking the integral (5.57) into two parts, over the subintervals $0 \leq \xi \leq x$ and $x \leq \xi \leq 1$, we arrive at the explicit solution formula

$$u(x) = \frac{1}{c} \int_0^x (1 - \xi) f(\xi) \, d\xi + \frac{1}{c} \int_x^1 x (1 - \xi) f(\xi) \, d\xi. \quad (5.58)$$

For example, under a constant unit force $f$, (5.58) yields the solution

$$u(x) = \frac{f}{c} \int_0^x (1 - \xi) \, d\xi + \frac{f}{c} \int_x^1 x (1 - \xi) \, d\xi = \frac{f}{2c} (1-x)x^2 + \frac{f}{2c} x (1-x)^2 = \frac{f}{2c} (x-x^2).$$
Let us, finally, convince ourselves that the superposition formula (5.58) does indeed give
the correct answer. First,
\[
c \frac{du}{dx} = (1 - x) f(x) + \int_0^x [-\xi f(\xi)] d\xi - x(1 - x) f(x) + \int_x^1 (1 - \xi) f(\xi) d\xi
\]
\[
= -\int_0^1 \xi f(\xi) d\xi + \int_x^1 f(\xi) d\xi.
\]
Differentiating again, we conclude that
\[
-c \frac{d^2 u}{dx^2} = f(x),
\]
as claimed.

**Remark:** In computing the derivatives of \(u\), we made use of the calculus formula
\[
\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} F(x, \xi) \, d\xi = F(x, \beta(x)) \frac{d\beta}{dx} - F(x, \alpha(x)) \frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}(x, \xi) \, d\xi
\]
(5.59)
for the derivative of an integral with variable limits — which is a straightforward conse-
quence of the Fundamental Theorem of Calculus and the chain rule, \([8, 123]\). As always,
one must be careful when interchanging the order of differentiation and integration.

Although this relatively simple problem is perhaps easier to solve directly, the Green’s
function approach helps crystallize our understanding, and provides a unified framework
that covers the full range of linear boundary value problems arising in applications, in-
cluding those governed by partial differential equations.

We note the following fundamental properties, that serve to uniquely characterize the
Green’s function. First, since the delta forcing vanishes except at the point \(x = \xi\), the
Green’s function satisfies the homogeneous differential equation\(†\)
\[
-c \frac{\partial^2 G}{\partial x^2} (x; \xi) = 0 \quad \text{for all} \quad x \neq \xi.
\]
(5.60)
Secondly, by construction, it must satisfy the boundary conditions,
\[
G(0; \xi) = 0 = G(1; \xi).
\]
Thirdly, for each fixed \(\xi\), \(G(x; \xi)\) is a continuous function of \(x\), but its derivative \(\partial G/\partial x\)
has a jump discontinuity of magnitude \(-1/c\) at the impulse point \(x = \xi\). As a result, the
second derivative \(\partial^2 G/\partial x^2\) has a delta function discontinuity there, and thereby solves the
original impulse boundary value problem (5.53).

Finally, we cannot help but notice that the Green’s function is a symmetric function of
its two arguments: \(G(x; \xi) = G(\xi; x)\). Symmetry has the interesting physical consequence
that the displacement of the bar at position \(x\) due to an impulse force concentrated at
position \(\xi\) is exactly the same as the displacement of the bar at \(\xi\) due to an impulse of the
same magnitude being applied at \(x\). This turns out to be a rather general, although perhaps

\(†\) Since \(G(x; \xi)\) is a function of two variables, we switch to partial derivative notation to indicate its derivatives.
unanticipated phenomenon. Symmetry is a consequence of the underlying symmetry or “self-adjointness” of the boundary value problem; this aspect will be developed in detail in Section 9.2.

**Example 5.10.** Let \( \omega^2 > 0 \) be a fixed positive constant. Let us solve the boundary value problem

\[
-u'' + \omega^2 u = f(x), \quad u(0) = u(1) = 0.
\]  

(5.61)

by constructing its Green’s function. To this end, we first consider the effect of a delta function inhomogeneity

\[
-u'' + \omega^2 u = \delta(x - \xi), \quad u(0) = u(1) = 0.
\]

(5.62)

Rather than try to integrate this differential equation directly, let us appeal to the defining properties of the Green’s function. The general solution to the homogeneous equation is a linear combination of the two basic exponentials \( e^{\omega x} \) and \( e^{-\omega x} \), or better, the hyperbolic functions

\[
\cosh \omega x = \frac{e^{\omega x} + e^{-\omega x}}{2}, \quad \sinh \omega x = \frac{e^{\omega x} - e^{-\omega x}}{2}.
\]

(5.63)

The solutions satisfying the first boundary condition are multiples of \( \sinh \omega x \), while those satisfying the second boundary condition are multiples of \( \sinh \omega (1 - x) \). Therefore, the solution to (5.62) has the form

\[
G(x; \xi) = \begin{cases} 
    a \sinh \omega x, & x < \xi, \\
    b \sinh \omega (1 - x), & x > \xi.
\end{cases}
\]

Continuity of \( G(x; \xi) \) at \( x = \xi \) requires

\[
a \sinh \omega \xi = b \sinh \omega (1 - \xi).
\]

(5.64)

At \( x = y \), the derivative \( \partial G/\partial x \) must have a jump discontinuity of magnitude \(-1\) in order that the second derivative term in (5.62) match the delta function. (The term \( \lambda u \) clearly can’t produce the required singularity.) Since

\[
\frac{\partial G}{\partial x}(x; \xi) = \begin{cases} 
    a \omega \cosh \omega x, & x < \xi, \\
    -b \omega \cosh \omega (1 - x), & x > \xi,
\end{cases}
\]

the jump condition requires

\[
a \omega \cosh \omega \xi - 1 = -b \omega \cosh \omega (1 - \xi).
\]

(5.65)

Multiplying (5.64) by \( \omega \cosh \omega (1 - \xi) \) and (5.65) by \( \sinh \omega (1 - \xi) \), and then adding the results together, we find

\[
\sinh \omega (1 - \xi) = a \omega \left[ \sinh \omega \xi \cosh \omega (1 - \xi) + \cosh \omega \xi \sinh \omega (1 - \xi) \right] = a \omega \sinh \omega,
\]

where we made use of the addition formula for the hyperbolic sine:

\[
\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta,
\]

(5.67)
which you are asked to prove in Exercise 1. Therefore, solving (5.65–66) for

\[ a = \frac{\sinh \omega (1 - \xi)}{\omega \sinh \omega}, \quad b = \frac{\sinh \omega \xi}{\omega \sinh \omega}, \]

we find the explicit formula

\[ G(x; \xi) = \begin{cases} \frac{\sinh \omega x \sinh \omega (1 - \xi)}{\omega \sinh \omega}, & x < \xi, \\ \frac{\sinh \omega (1 - x) \sinh \omega \xi}{\omega \sinh \omega}, & x > \xi, \end{cases} \tag{5.68} \]

for the Green’s function. A graph appears in Figure 5.9; note that the corner, indicating a discontinuity in the first derivative, appears at the point \( x = \xi \) where the impulse force is applied. As in the previous example, \( G(x; \xi) = G(\xi; x) \) is a symmetric function.

The general solution to the inhomogeneous boundary value problem (5.61) is given by the corresponding superposition formula (5.57), which becomes

\[
    u(x) = \int_0^1 G(x; \xi) f(\xi) \, d\xi \\
    = \int_0^x \frac{\sinh \omega (1 - x) \sinh \omega \xi}{\omega \sinh \omega} f(\xi) \, d\xi + \int_x^1 \frac{\sinh \omega x \sinh \omega (1 - \xi)}{\omega \sinh \omega} f(\xi) \, d\xi.
\]

For example, under a constant unit force \( f(x) \equiv 1 \), the solution is

\[
    u(x) = \int_0^x \frac{\sinh \omega (1 - x) \sinh \omega y}{\omega \sinh \omega} \, dy + \int_x^1 \frac{\sinh \omega x \sinh \omega (1 - \xi)}{\omega \sinh \omega} \, d\xi \\
    = \frac{\sinh \omega (1 - x) \left( \cosh \omega x - 1 \right)}{\omega^2 \sinh \omega} + \frac{\sinh \omega x \left( \cosh \omega (1 - x) - 1 \right)}{\omega^2 \sinh \omega} \tag{5.69} \\
    = \frac{1}{\omega^2} - \frac{\sinh \omega x + \sinh \omega (1 - x)}{\omega^2 \sinh \omega}.
\]

For comparative purposes, the reader may wish to rederive this particular solution by a direct calculation, without appealing to the Green’s function.
Example 5.11. Finally, consider the Neumann boundary value problem

$$-c u'' = f(x), \quad u'(0) = 0 = u'(1),$$

modeling the equilibrium deformation a homogeneous bar with two free ends when subject to an external force $f(x)$. The Green’s function should satisfy the particular case

$$-c u'' = \delta(x - \xi), \quad u'(0) = 0 = u'(1),$$

when the forcing function is a concentrated impulse. As in Example 5.9, the general solution to the latter differential equation is

$$u(x) = -\frac{\rho(x - \xi)}{c} + ax + b,$$

where $a, b$ are integration constants, and $\rho$ is the ramp function (5.25). However, the Neumann boundary conditions require that

$$u'(0) = a = 0, \quad u'(1) = -\frac{1}{c} + a = 0,$$

which cannot both be satisfied. We are forced to conclude that there is no Green’s function in this case.

The underlying problem is that the Neumann boundary value problem (5.70) does not admit a unique solution, and hence cannot admit a Green’s function solution formula (5.57). Indeed, integrating twice, we find that the general solution to the differential equation is

$$u(x) = a x + b - \frac{1}{c} \int_0^x \int_0^y f(z) \, dz \, dy,$$

where $a, b$ are integration constants. Since

$$u'(x) = a - \frac{1}{c} \int_0^x f(z) \, dz,$$

the boundary conditions require that

$$u'(0) = a = 0, \quad u'(1) = a - \frac{1}{c} \int_0^1 f(z) \, dz = 0.$$

These equations are compatible if and only if

$$\int_0^1 f(z) \, dz = 0.$$  \hfill (5.71)

Thus, the Neumann boundary value problem admits a solution if and only if there is no net force on the bar. Indeed, physically, if (5.71) does not hold, then, because its ends are not attached to any support, the bar cannot stay in equilibrium, but will move off in the direction of the net force. On the other hand, if (5.71) holds, then the solution

$$u(x) = b - \frac{1}{c} \int_0^x \int_0^y f(z) \, dz \, dy$$
is not unique, since the integration constant $b$ is not constrained by the boundary conditions, and so can assume any constant value. Physically, this means that we can translate any equilibrium configuration of the bar to obtain another valid equilibrium.

The non-existence of the Green’s function stems from the fact that the boundary value problem does not admit a unique solution; it either has no solution or infinitely many solutions. This should remind the reader of a finite-dimensional linear system; this point is developed in detail in [104].

**Remark:** The constraint (5.71) is a manifestation of the Fredholm alternative, to be developed in detail in Section 9.1.

Let us summarize the fundamental properties that serve to characterize the Green’s function, in a form that applies to general second order boundary value problems governed by a regular differential equation

$$p(x) \frac{d^2u}{dx^2} + q(x) \frac{du}{dx} + r(x) u(x) = f(x)$$  \hspace{1cm} (5.72)

with continuous coefficients, $p, q, r, f \in C^0[a,b]$, with $p(x) \neq 0$ for all $a \leq x \leq b$, combined with a pair of homogeneous boundary conditions at the ends.

**Basic Properties of the Green’s Function $G(x; \xi)$**

(i) Solves the homogeneous differential equation at all points $x \neq \xi$.

(ii) Satisfies the homogeneous boundary conditions.

(iii) Is a continuous function of its arguments.

(iv) For each fixed $\xi$, its derivative $\partial G/\partial x$ is piecewise $C^1$, with a single jump discontinuity of magnitude $1/p(\xi)$ at the impulse point $x = \xi$.

With the Green’s function in hand, the solution to the general boundary value problem (5.72) subject to the appropriate homogeneous boundary conditions is given by the Superposition Formula

$$u(x) = \int_a^b G(x; \xi) f(\xi) d\xi.$$  \hspace{1cm} (5.73)

The symmetry of the Green’s function is more subtle, and relies on the self-adjointness of the boundary value problem, an issue to be addressed in detail in Chapter 9. In the present situation, (5.72) turns out to define a self-adjoint problem if and only if $q(x) = p'(x)$; in this case $G(\xi; x) = G(x; \xi)$.

Finally, as we saw in Example 5.11, not every such boundary value problem admits a solution, and one expects to find a Green’s function only in cases in which the solution exists and is unique. See [23] for a proof of the following general result.

**Theorem 5.12.** The following are equivalent:

(a) The only solution to homogeneous boundary value problem with $f(x) \equiv 0$ is the zero function $u(x) \equiv 0$.

(b) The inhomogeneous boundary value problem has a unique solution for every choice of forcing function.

(c) The boundary value problem admits a Green’s function.
5.3. The Green’s Function for the Poisson Equation.

Now we turn to develop the Green’s function approach for solving boundary value problems involving the two-dimensional Poisson equation (4.73). As before, the Green’s function is characterized as the solution to the homogeneous boundary value problem in which the inhomogeneity is a concentrated unit impulse — a delta function. The solution to the general forced boundary value problem is then obtained via linear superposition, that is, as a convolution integral with the Green’s function.

*Calculus in Two Dimensions*

Before proceeding, we need to quickly review some basic facts concerning vector calculus in the plane. The student may wish to consult a standard multivariable calculus text, e.g., [8, 123], for the full details.

Let \( \mathbf{x} = (x, y) \) denote the usual Cartesian coordinates on the plane \( \mathbb{R}^2 \). The term *scalar field* is synonymous with a real-valued function \( u(x, y) \), defined on a domain \( \Omega \subset \mathbb{R}^2 \). A vector-valued function\(^\dagger\)

\[
\mathbf{v}(\mathbf{x}) = \mathbf{v}(x, y) = (v_1(x, y), v_2(x, y))
\]

is known as a (planar) *vector field*. A vector field assigns a vector \( \mathbf{v}(x, y) \) to each point \( (x, y) \) in its domain of definition, and hence defines a function \( \mathbf{v} : \Omega \to \mathbb{R}^2 \). Physical examples include velocity vector fields of fluid flows, heat flux vector fields in thermodynamics, and gravitational and electrostatic force vector fields.

The *gradient* operator \( \nabla \) maps the scalar field \( u(x, y) \) to the vector field

\[
\nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right).
\]

The scalar field \( u \) is often referred to as a *potential function* for its gradient vector field \( \mathbf{v} = \nabla u \). On a connected domain \( \Omega \), the potential, when it exists, is uniquely determined up to addition of a constant.

The *divergence* of the planar vector field \( \mathbf{v} = (v_1, v_2) \) is the scalar field

\[
\nabla \cdot \mathbf{v} = \text{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}.
\]

Its *curl* is defined as

\[
\nabla \times \mathbf{v} = \text{curl} \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.
\]

Notice that the curl of a planar vector field is also a scalar field. (In contrast, in three dimensions, the curl of a vector field is another vector field.) We note that, for a sufficiently smooth potential \( u \in C^2 \), the curl of its gradient vector field automatically vanishes:

\[
\nabla \times \nabla u = \frac{\partial}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial}{\partial y} \frac{\partial u}{\partial x} \equiv 0,
\]

\(\dagger\) For typographical reasons, we will forgo using column vector notation in this section.
by the equality of mixed partials. Thus, a necessary condition for a vector field $\mathbf{v}$ to admit a potential is that it is \textit{irrotational}, meaning $\nabla \times \mathbf{v} = 0$; this condition is sufficient if the underlying domain $\Omega$ is \textit{simply connected}, i.e., has no holes, [8]. On the other hand, the divergence of a gradient vector field coincides with the Laplacian of the potential function:

$$\nabla \cdot \nabla u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{5.78}$$

A vector field is \textit{incompressible} if it has zero divergence: $\nabla \cdot \mathbf{v} = 0$; for the velocity vector field of a steady state fluid flow, incompressibility means that the fluid does not change volume. Water is, for all practical purposes, an incompressible fluid. Therefore, an irrotational vector field with potential $u$ is also incompressible if and only if the potential solves the Laplace equation $\Delta u = 0$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary $\partial \Omega$ consists of one or more piecewise smooth closed curves. We orient the boundary in the usual, counterclockwise direction, or, more precisely, so that the domain is always on one’s left as one goes around the bounding curve(s). Figure 5.10 sketches a domain with two holes; its three boundary curves are oriented according to the directions of the arrows. Note that the outer boundary curve is traversed in a counterclockwise direction, while the two inner boundary curves are traversed clockwise.

\textit{Green’s Theorem}, first formulated by George Green to use in his seminal study of partial differential equations and potential theory, relates certain double integrals over the domain to line integrals around its boundary. It should be viewed as the extension of the Fundamental Theorem of Calculus to double integrals. (Indeed, the standard proofs, [8], reduce it to the one variable theorem.)

\textbf{Theorem 5.13.} Let $\mathbf{v}(\mathbf{x})$ be a smooth\footnote{To be precise, we require $\mathbf{v}$ to be continuously differentiable within the domain, and continuous up to the boundary, so $\mathbf{v} \in C^0(\overline{\Omega}) \cap C^1(\Omega)$, where $\overline{\Omega} = \Omega \cup \partial \Omega$ denotes the closure of the domain $\Omega$.} vector field defined on a bounded domain $\Omega \subset \mathbb{R}^2$. Then the line integral of $\mathbf{v}$ around the boundary $\partial \Omega$ equals the double integral...
of its curl over the domain:

$$\iint_{\Omega} \nabla \times \mathbf{v} \, dx \, dy = \oint_{\partial \Omega} \mathbf{v} \cdot d\mathbf{x}, \quad (5.79)$$

or, in full detail,

$$\iint_{\Omega} \left( \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \, dx \, dy = \oint_{\partial \Omega} v_1 \, dx + v_2 \, dy. \quad (5.80)$$

**Example 5.14.** Let us apply Green’s Theorem 5.13 to the particular vector field $\mathbf{v} = (0, x)$. Since $\nabla \times \mathbf{v} \equiv 1$, we find

$$\oint_{\partial \Omega} x \, dy = \iint_{\Omega} dx \, dy = \text{area } \Omega. \quad (5.81)$$

This means that we can compute the area of a planar domain by evaluating the indicated line integral around its boundary.

For our purposes, we need to rewrite the basic Green identity (5.79) in an equivalent “divergence form”. Given a planar vector field $\mathbf{v} = (v_1, v_2)$, let

$$\mathbf{v}^\perp = (-v_2, v_1) \quad (5.82)$$
denote the “perpendicular” vector field. We note that its curl

$$\nabla \times \mathbf{v}^\perp = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \nabla \cdot \mathbf{v}, \quad (5.83)$$

coincides with the divergence of the original vector field.

When we replace $\mathbf{v}$ in Green’s identity (5.79) by $\mathbf{v}^\perp$, the result is

$$\iint_{\Omega} \nabla \cdot \mathbf{v} \, dx \, dy = \iint_{\Omega} \nabla \times \mathbf{v}^\perp \, dx \, dy = \oint_{\partial \Omega} \mathbf{v}^\perp \cdot d\mathbf{x} = \oint_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ denotes the unit outwards normal to the boundary of our domain, while $ds$ denotes the arc length element along the boundary curve. This yields the divergence form of Green’s Theorem:

$$\iint_{\Omega} \nabla \cdot \mathbf{v} \, dx \, dy = \oint_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \, ds. \quad (5.84)$$

Physically, if $\mathbf{v}$ represents the velocity vector field of a steady state fluid flow, then the line integral in (5.84) represents the net fluid flux out of the region $\Omega$. As a result, the divergence $\nabla \cdot \mathbf{v}$ represents the local change in area of the fluid at each point, which serves to justify our earlier statement on incompressibility.

Consider next the product vector field $u \mathbf{v}$ obtained by multiplying a scalar field $u$ by a vector field $\mathbf{v}$. An elementary computation proves that its divergence is

$$\nabla \cdot (u \mathbf{v}) = u \nabla \cdot \mathbf{v} + \nabla u \cdot \mathbf{v}. \quad (5.85)$$
Replacing \( \mathbf{v} \) by \( \mathbf{u} \mathbf{v} \) in the divergence formula (5.84), we deduce what is usually referred to as Green’s formula

\[
\int \int_{\Omega} \left( \mathbf{u} \nabla \cdot \mathbf{v} + \nabla \mathbf{u} \cdot \mathbf{v} \right) dx dy = \int_{\partial \Omega} \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{n}) ds,
\]

which is valid for arbitrary bounded domains \( \Omega \), and arbitrary \( C^1 \) scalar and vector fields defined thereon. Rearranging the terms produces

\[
\int \int_{\Omega} \nabla \mathbf{u} \cdot \mathbf{v} dx dy = \oint_{\partial \Omega} \mathbf{u} \left( \mathbf{v} \cdot \mathbf{n} \right) ds - \int \int_{\Omega} \mathbf{u} \nabla \cdot \mathbf{v} dx dy.
\]

We will use this identity as an integration by parts formula for double integrals. Indeed, comparing with the one-dimensional integration by parts formula

\[
\int_{a}^{b} u' v(x) dx = u(b)v(b) - \int_{a}^{b} u(x)v'(x) dx,
\]

the single integrals have become double integrals; the derivatives are vector derivatives — gradient and divergence; while the contribution at the boundary or endpoints of the interval is replaced by a line integral around the entire boundary of the two-dimensional domain.

A useful special case of (5.86) is when \( \mathbf{v} = \nabla v \) is the gradient of a scalar field \( v \). Then, in view of (5.78), Green’s formula (5.86) becomes

\[
\int \int_{\Omega} \left( u \Delta v + \nabla u \cdot \nabla v \right) dx dy = \oint_{\partial \Omega} u \frac{\partial v}{\partial \mathbf{n}} ds,
\]

where \( \frac{\partial v}{\partial \mathbf{n}} = \nabla v \cdot \mathbf{n} \) is the normal derivative of the scalar field \( v \) on the boundary of the domain. In particular, setting \( v = u \), we deduce

\[
\int \int_{\Omega} \left( u \Delta u + \| \nabla u \|^2 \right) dx dy = \oint_{\partial \Omega} u \frac{\partial u}{\partial \mathbf{n}} ds.
\]

As an application, we establish the basic uniqueness result for solutions to the boundary value problems for the Poisson equation:

**Theorem 5.15.** Suppose \( \tilde{u} \) and \( u \) both satisfy the same inhomogeneous Dirichlet or mixed boundary value problem for the Poisson equation on a connected, bounded domain \( \Omega \). Then \( \tilde{u} = u \). On the other hand, if \( \tilde{u} \) and \( u \) satisfy the same Neumann boundary value problem, then \( \tilde{u} = u + c \) for some constant \( c \).

**Proof:** Since, by assumption, \( -\Delta \tilde{u} = f = -\Delta u \), the difference \( v = \tilde{u} - u \) satisfies the Laplace equation \( \Delta v = 0 \) in \( \Omega \). Similarly, the difference \( v \) has homogeneous boundary conditions. Therefore, applying (5.90) to \( v \), we find

\[
\int \int_{\Omega} \| \nabla v \|^2 dx dy = \oint_{\partial \Omega} v \frac{\partial v}{\partial \mathbf{n}} ds = 0,
\]

since, at every point on the boundary, either \( v = 0 \) or \( \partial v / \partial \mathbf{n} = 0 \). Since the integrand is continuous and everywhere non-negative, we immediately conclude that \( \| \nabla v \|^2 = 0 \), and hence \( \nabla v = \mathbf{0} \) throughout \( \Omega \). On a connected domain, the only functions annihilated by the gradient operator are the constants:
Lemma 5.16. If \( v(x, y) \) is a \( C^1 \) function defined on a connected domain \( \Omega \subset \mathbb{R}^2 \), then \( \nabla v \equiv 0 \) if and only if \( v(x, y) \equiv c \) is a constant.

Proof: Let \( a, b \) are any two points in \( \Omega \). Then, by connectivity, we can find a curve \( C \) connecting them. The Fundamental Theorem for line integrals, \([8, 123]\), states that

\[
\int_C \nabla u \cdot dx = u(b) - u(a).
\]

Thus, if \( \nabla u \equiv 0 \), then \( u(b) = u(a) \), which shows that \( u \) is constant. \( Q.E.D. \)

We conclude that \( \tilde{u} = u + v = u + c \), which proves the result in the Neumann case. In the Dirichlet or mixed problems, there is at least one point on the boundary where \( v = 0 \), and hence the only possible constant is \( v = c = 0 \), proving that \( \tilde{u} = u \). \( Q.E.D. \)

Thus, the Dirichlet and mixed boundary value problems admit at most one solution, while the Neumann boundary value problem has either no solutions or infinitely many solutions. Proof of existence of solutions is more challenging, and will be left to a more advanced text, \([36, 73, 87]\).

If we subtract from formula (5.89) the formula

\[
\iint_{\Omega} (v \Delta u + \nabla u \cdot \nabla v) \, dx \, dy = \oint_{\partial \Omega} v \frac{\partial u}{\partial n} \, ds,
\]

obtained by interchanging \( u \) and \( v \), we obtain the identity

\[
\iint_{\Omega} (u \Delta v - v \Delta u) \, dx \, dy = \oint_{\partial \Omega} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, ds,
\]

which will play an essential role in our analysis of the Poisson equation.

Setting \( v = 1 \) in (5.91) yields

\[
\iint_{\Omega} \Delta u \, dx \, dy = \oint_{\partial \Omega} \frac{\partial u}{\partial n} \, ds.
\]

Suppose \( u \) solves the Neumann boundary value problem

\[- \Delta u = f, \quad \text{in} \quad \Omega \quad \frac{\partial u}{\partial n} = h \quad \text{on} \quad \partial \Omega.\]

Then (5.93) requires that

\[
\iint_{\Omega} f \, dx \, dy + \oint_{\partial \Omega} h \, ds = 0,
\]

which thus forms a necessary condition for the existence of a solution \( u \). Physically, if \( u \) represents the equilibrium temperature of a plate, then the integrals in (5.93) measure the net loss in heat energy due to, respectively, the external heat source and the heat flux through the boundary. Equation (5.94) says that, to remain in thermal equilibrium, there can be no net loss (or gain) in the plate’s heat energy over time.
The Two–Dimensional Delta Function

Now, let us return to business at hand — solving the Poisson equation on a bounded domain \( \Omega \subset \mathbb{R}^2 \). We will subject the solution to either homogeneous Dirichlet boundary conditions or homogeneous mixed boundary conditions. (As we just noted, the Neumann boundary value problem does not admit a unique solution, and hence does not possess a Green’s function.) The Green’s function for the boundary value problem arises when the forcing function is a unit impulse concentrated at a single point in the domain.

Thus, our first task is to establish the proper form for a unit impulse in our two-dimensional context. The \textit{delta function} concentrated at a point \( \xi = (\xi, \eta) \in \mathbb{R}^2 \) is denoted by
\[
\delta_{\xi}(x, y) = \delta(\xi, \eta)(x, y) = \delta(x - \xi, y - \eta),
\]
and is designed so that
\[
\delta_{\xi}(x) = 0, \quad x \neq \xi, \quad \int \int_{\Omega} \delta_{(\xi, \eta)}(x, y) \, dx \, dy = 1, \quad \xi \in \Omega.
\]
In particular, \( \delta(x, y) = \delta_0(x, y) \) represents the delta function at the origin. As in the one-dimensional version, there is no ordinary function that satisfies both criteria; rather, \( \delta(x, y) \) is to be viewed as the limit of a sequence of more and more highly concentrated functions \( g_n(x, y) \), with
\[
\lim_{n \to \infty} g_n(x, y) = 0, \quad \text{for} \quad (x, y) \neq (0, 0), \quad \text{while} \quad \int \int_{\mathbb{R}^2} g_n(x, y) \, dx \, dy = 1.
\]
A good example of a suitable sequence is provided by the \textit{radial Gaussian functions}
\[
g_n(x, y) = \frac{n}{\pi} e^{-n(x^2 + y^2)}.
\]
As plotted in Figure 5.11, as \( n \to \infty \), the Gaussian profiles become more and more concentrated near the origin, while maintaining a unit volume underneath their graphs. The fact that their integral over \( \mathbb{R}^2 \) equals 1 is a consequence of \( (\text{Gaussint}^2) \).

Alternatively, one can assign the delta function a dual interpretation as the linear functional
\[
L_{(\xi, \eta)}[u] = L_{\xi}[u] = u(\xi) = u(\xi, \eta),
\]
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that assigns to each continuous function \( u \in C^0(\Omega) \) its value at the point \( \xi \in \Omega \). Then, using the \( L^2 \) inner product
\[
\langle u; v \rangle = \int\int_{\Omega} u(x, y) v(x, y) \, dx \, dy.
\] between scalar fields \( u, v \in C^0(\Omega) \), we formally identify the linear functional \( L_\xi \) with the delta “function” by the integral formula
\[
\langle \delta(\xi, \eta); u \rangle = \int\int_{\Omega} \delta(\xi, \eta)(x, y) u(x, y) \, dx \, dy = \left\{ \begin{array}{ll}
u(\xi, \eta), & (\xi, \eta) \in \Omega, \\
0, & (\xi, \eta) \in \mathbb{R}^2 \setminus \Omega,
\end{array} \right.
\] for any \( u \in C^0(\overline{\Omega}) \). As in the one-dimensional version, we will avoid defining the integral when the delta function is concentrated at a boundary point of the domain.

Since double integrals can be evaluated as repeated one-dimensional integrals, we can conveniently view
\[
\delta(\xi, \eta)(x, y) = \delta(x - \xi) \delta(y - \eta)
\] as the product† of a pair of one-dimensional delta functions. Indeed, if
\[
(\xi, \eta) \in R = \{a < x < b, c < y < d\} \subset \Omega
\] is contained in a rectangle that lies within the domain, then
\[
\int\int_{\Omega} \delta(\xi, \eta)(x, y) u(x, y) \, dx \, dy = \int\int_{R} \delta(\xi, \eta)(x, y) u(x, y) \, dx \, dy
\]
\[
= \int_a^b \left( \int_c^d \delta(x - \xi) \delta(y - \eta) u(x, y) \, dy \right) \, dx = \int_a^b \delta(x - \xi) u(x, \eta) \, dx = u(\xi, \eta).
\]

The Green’s Function

As in the one-dimensional context, the Green’s function is defined as the solution to the inhomogeneous differential equation when subject to a concentrated unit delta force at a prescribed point \( \xi = (\xi, \eta) \in \Omega \) inside the domain. In the case of Poisson’s equation, the partial differential equation takes the form
\[
- \Delta u = \delta_\xi, \quad \text{or, explicitly,} \quad - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \delta(x - \xi) \delta(y - \eta),
\] and the solution is subject to homogeneous boundary conditions, say the Dirichlet conditions \( u = 0 \) on \( \partial \Omega \). The solution to the resulting boundary value problem is the Green’s function, and written
\[
G_\xi(x) = G(x; \xi) = G(x, y; \xi, \eta).
\]† This is an exception to our earlier injunction not to multiply delta functions. Multiplication is allowed if they depend on different variables.

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Once we know the Green’s function, the solution to the general Poisson boundary value problem
\[-\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \quad (5.104)\]
is reconstructed through a superposition principle. We regard the forcing function
\[f(x, y) = \int\int_\Omega \delta(x - \xi) \delta(y - \eta) f(\xi, \eta) \, d\xi \, d\eta\]
as a superposition of delta impulses, whose strength equals the value of \(f\) at the impulse point. Linearity implies that the solution to the boundary value problem is the corresponding superposition of Green’s function responses to each of the constituent impulses. The net result is the fundamental superposition formula
\[u(x, y) = \int\int_\Omega G(x, y; \xi, \eta) f(\xi, \eta) \, d\xi \, d\eta. \quad (5.105)\]
for the solution. This can be verified by direct evaluation:
\[-\Delta u(x, y) = \int\int_\Omega \left[-\Delta G(x, y; \xi, \eta) \right] f(\xi, \eta) \, d\xi \, d\eta
= \int\int_\Omega \delta(x - \xi, y - \eta) f(\xi, \eta) \, d\xi \, d\eta = f(x, y),\]
as claimed. Also, when \((x, y) \in \partial\Omega\), we have \(u(x, y) = 0\) since \(G(x, y; \xi, \eta) = 0\) for all \(\xi, \eta\).

The Green’s function turns out to be symmetric under interchange of its arguments:
\[G(\xi, \eta; x, y) = G(x, y; \xi, \eta). \quad (5.106)\]
As in the one-dimensional case, symmetry is a consequence of the self-adjointness of the boundary value problem, and will be proved in general in Chapter 9. Symmetry has the following intriguing physical interpretation: Let \(x, \xi \in \Omega\) be any two points in the domain. We apply a unit impulse to the membrane at the first point, and measure its deflection at the second; the result is exactly the same as if we apply the impulse at the second point, and measure the deflection at the first! (Deflections at other points in the domain will typically bear very little connection with each other.) Similarly, in electrostatics, the solution \(u(x, y)\) is interpreted as the electrostatic potential for a system of charges in equilibrium. A delta function corresponds to a point charge, e.g., an electron. The symmetry property says that the electrostatic potential at \(x\) due to a point charge placed at position \(\xi\) is exactly the same as the potential at \(\xi\) due to a point charge at \(x\). The reader may wish to meditate on the physical plausibility of these striking facts.

Unfortunately, most Green’s functions cannot be written down in closed form. One important exception is when the domain is the entire plane: \(\Omega = \mathbb{R}^2\). The solution to the Poisson equation \((5.102)\) is the free space Green’s function \(G_0(x, y; \xi, \eta) = G_0(x, \xi)\) which measures the effect of a concentrated unit impulse throughout two-dimensional space, e.g., the gravitational potential due to a point mass or the electrostatic potential due to a point charge. To motivate the construction, let us appeal to physical intuition. First, since the impulse is concentrated at \(x = \xi\), the function must solve the homogeneous Laplace
equation $\Delta G_0 = 0$ except at $x = \xi$, where we expect it to have some sort of discontinuity. Second, since the Poisson equation is modeling a homogeneous, uniform medium, in the absence of boundary conditions the effect of a unit impulse should only depend upon on the distance away from its source. Therefore, we expect $G_0$ to only depend on the radial variable:

$$G_0 = v(r), \text{ where } r = \| x - \xi \| = \sqrt{(x - \xi)^2 + (y - \eta)^2}.$$ 

According to (4.103), the only radially symmetric solutions to the Laplace equation are

$$v(r) = a + b \log r,$$

where $a$ and $b$ are constants. The constant term $a$ has zero derivative, and so cannot contribute to the delta function singularity. Therefore, we expect the required solution to be a multiple of the logarithmic term. The correct multiple, setting

$$G_0(x, y; \xi, \eta) = -\frac{1}{2\pi} \log r = -\frac{1}{2\pi} \log \| x - \xi \| = -\frac{1}{4\pi} \log \left[ (x - \xi)^2 + (y - \eta)^2 \right],$$

is dictated by the following result, known as Green’s representation formula.

**Lemma 5.17.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, with piecewise $C^1$ boundary $\partial \Omega$. Suppose $u \in C^2(\Omega) \cap C^1(\Omega)$. Then, for any $(x, y) \in \Omega$,

$$u(x, y) = -\int_{\Omega} G_0(x, y; \xi, \eta) \Delta u(\xi, \eta) d\xi d\eta +$$

$$+ \oint_{\partial \Omega} \left( G_0(x, y; \xi, \eta) \frac{\partial u}{\partial n}(\xi, \eta) - \frac{\partial G_0}{\partial n}(x, y; \xi, \eta) u(\xi, \eta) \right) ds,$$

where the normal derivatives on the boundary are taken with respect to the integration variables $\xi = (\xi, \eta)$.

**Proof:** We first note that, even though $G_0(x, \xi)$ has a logarithmic singularity at $x = \xi$, the double integral in (5.109) is well defined. Indeed, rewriting it in polar coordinates $\xi = x + r \cos \theta$, $\eta = y + r \sin \theta$, and recalling $d\xi d\eta = r dr d\theta$, it equals

$$\frac{1}{4\pi} \int \int (r \log r) \Delta u dr d\theta.$$ 

The product $r \log r$ is everywhere continuous — even at $r = 0$ — and so the integral exists and is finite. There is, of course, no problem with the line integral in (5.109) since the contour does not go through the singularity.

By construction,

$$-\Delta G_0 = 0 \quad \text{for all} \quad x \neq \xi,$$

where the Laplacian can be taken either with respect to $x$ or with respect to $\xi$, since the variables enter symmetrically into the formula for $G_0(x; \xi) = G_0(\xi; x)$. Thus, if there were no singularity, the Green formula (5.92) would immediately imply the vanishing of
the right hand side of (5.109), which would be wrong. Rather, we will apply the Green formula, but avoid the singularity by working on a subdomain

$$\Omega_\varepsilon = \Omega \setminus D_\varepsilon(x) = \{ \xi \in \Omega \mid \| x - \xi \| > \varepsilon \}$$

obtained by cutting out a small disk

$$D_\varepsilon(x) = \{ \xi \mid \| x - \xi \| \leq \varepsilon \}$$

of radius $\varepsilon > 0$ centered at $x$; the subdomain $\Omega_\varepsilon$ is represented by the shaded region in Figure 5.12. We choose $\varepsilon$ sufficiently small so that $D_\varepsilon(x) \subset \Omega$, and hence

$$\partial \Omega_\varepsilon = \partial \Omega \cup C_\varepsilon,$$

where

$$C_\varepsilon = \{ \| x - \xi \| = \varepsilon \}$$

is the circular boundary of the disk. Since the double integral is well-defined, we can approximate it by integrating over $\Omega_\varepsilon$:

$$\int \int_{\Omega_\varepsilon} G_0(x, y; \xi, \eta) \Delta u(\xi, \eta) \, d\xi \, d\eta = \lim_{\varepsilon \to 0} \int \int_{\Omega_\varepsilon} G_0(x, y; \xi, \eta) \Delta u(\xi, \eta) \, d\xi \, d\eta. \quad (5.111)$$

Since $G_0$ has no singularities in $\Omega_\varepsilon$, we are able to apply the Green formula (5.89) and then (5.110) to evaluate

$$\int \int_{\Omega_\varepsilon} G_0(x, y; \xi, \eta) \Delta u(\xi, \eta) \, d\xi \, d\eta$$

$$= \oint_{\partial \Omega} \left( G_0(x, y; \xi, \eta) \frac{\partial u}{\partial n}(\xi, \eta) - \frac{\partial G_0}{\partial n}(x, y; \xi, \eta) u(\xi, \eta) \right) \, ds - \oint_{C_\varepsilon} \left( G_0(x, y; \xi, \eta) \frac{\partial u}{\partial n}(\xi, \eta) - \frac{\partial G_0}{\partial n}(x, y; \xi, \eta) u(\xi, \eta) \right) \, ds, \quad (5.112)$$

where the line integral around $C_\varepsilon$ is taken in the usual counterclockwise direction — the opposite orientation to that induced by its status as part of the boundary of $\Omega_\varepsilon$. Now, on the circle $C_\varepsilon$,

$$G_0(x, y; \xi, \eta) = - \log \frac{r}{2\pi} \bigg|_{r=\varepsilon} = - \log \frac{\varepsilon}{2\pi}, \quad (5.113)$$
while, in view of Exercise \( \blacksquare \),

\[
\frac{\partial G_0}{\partial n}(x, y; \xi, \eta) = -\frac{1}{2\pi} \frac{\partial (\log r)}{\partial r} \bigg|_{r=\varepsilon} = -\frac{1}{2\pi \varepsilon}.
\]  

(5.114)

Therefore,

\[
\oint_{C_\varepsilon} \frac{\partial G_0}{\partial n}(x, y; \xi, \eta) u(\xi, \eta) \, ds = -\frac{1}{2\pi \varepsilon} \oint_{C_\varepsilon} u(\xi, \eta) \, ds,
\]

which we recognize as minus the average of \( u \) on the circle of radius \( \varepsilon \). As \( \varepsilon \to 0 \), the circles shrink down to their common center, and so, by continuity, the averages tend to the value \( u(x, y) \) at the center:

\[
\lim_{\varepsilon \to 0} \oint_{C_\varepsilon} \frac{\partial G_0}{\partial n}(x, y; \xi, \eta) u(\xi, \eta) \, ds = -u(x, y).
\]  

(5.115)

On the other hand, using (5.113), and then (5.93) on the disk \( D_\varepsilon \):

\[
\oint_{C_\varepsilon} G_0(x, y; \xi, \eta) \frac{\partial u}{\partial n}(\xi, \eta) \, ds = -\frac{\log \varepsilon}{2\pi} \oint_{C_\varepsilon} \frac{\partial u}{\partial n}(\xi, \eta) \, ds
\]

\[
= -\frac{\log \varepsilon}{2\pi} \int \int_{D_\varepsilon} \Delta u(\xi, \eta) \, d\xi \, d\eta = -(\varepsilon^2 \log \varepsilon) \overline{\Delta u_\varepsilon},
\]

where

\[
\overline{\Delta u_\varepsilon} = \frac{1}{2\pi \varepsilon^2} \int \int_{D_\varepsilon} \Delta u(\xi, \eta) \, d\xi \, d\eta
\]

is the average of \( \Delta u \) over the disk \( D_\varepsilon \). As \( \varepsilon \to 0 \), the disks shrink to their common center, and so, by continuity, the averages tend to the value of \( \Delta u(x, y) \) at the center. Thus,

\[
\lim_{\varepsilon \to 0} \oint_{C_\varepsilon} G_0(x, y; \xi, \eta) \frac{\partial u}{\partial n}(\xi, \eta) \, ds = \lim_{\varepsilon \to 0} \left( -\varepsilon^2 \log \varepsilon \right) \overline{\Delta u_\varepsilon} = 0.
\]  

(5.116)

In view of (5.111, 115, 116), the \( \varepsilon \to 0 \) limit of (5.112) is exactly the Green representation formula (5.109).

**Q.E.D.**

**Corollary 5.18.** If \( u(x, y) \equiv 0 \) for all \( \|x\| > R \), then

\[
u(x, y) = -\int \int_{\mathbb{R}^2} G_0(x, y; \xi, \eta) \Delta u(\xi, \eta) \, d\xi \, d\eta.
\]  

(5.117)

This follows from (5.109) by taking \( \Omega \) to be any domain containing \( D_R = \{ \|x\| \leq R \} \), and so both \( u \) and \( \partial u/\partial n \) vanish on \( \partial \Omega \). If we formally apply the Green identity (5.92) to the double integral in (5.117), we obtain

\[
u(x, y) = \int \int_{\mathbb{R}^2} \left[ -\Delta G_0(x, y; \xi, \eta) \right] u(\xi, \eta) \, d\xi \, d\eta = \int \int_{\mathbb{R}^2} \delta(x - \xi) \delta(y - \eta) \, u(\xi, \eta) \, d\xi \, d\eta.
\]  

(5.118)

It is in this sense that we establish the key identity

\[-\Delta G_0(\xi, \xi) = \frac{1}{2\pi} \Delta (\log r) = \delta(x - \xi).
\]  

(5.119)
As noted above, the free space Green’s function (5.108) represents the gravitational potential in empty two-dimensional space due to a unit point mass, or, equivalently, the two-dimensional electrostatic potential due to a point charge, at position $\xi$. The corresponding gravitational (electrostatic) force field is obtained by taking its gradient:

$$\mathbf{F} = \nabla G_0 = -\frac{x - \xi}{2\pi \|x - \xi\|^2}.$$ 

Its magnitude

$$\|\mathbf{F}\| = \frac{1}{2\pi \|x - \xi\|}$$

is inversely proportional to the distance from the mass (charge), which is the two-dimensional form of Newton’s (Coulomb’s) three-dimensional inverse square law.

The gravitational potential due to a mass, e.g., a plate, in the shape of a domain $\Omega \subset \mathbb{R}^2$ is obtained by superimposing delta function sources with strengths equal to the density of the material at each point. The result is the potential function

$$u(x, y) = -\frac{1}{4\pi} \int_\Omega \rho(\xi, \eta) \log \left[ (x - \xi)^2 + (y - \eta)^2 \right] d\xi d\eta,$$  

(5.120)

in which $\rho(\xi, \eta)$ denotes the density at position $(\xi, \eta)$. For example, the gravitational potential due to the unit disk $D = \{ x^2 + y^2 \leq 1 \}$ with unit density $\rho \equiv 1$ is

$$u(x, y) = -\frac{1}{4\pi} \int_D \log \left[ (x - \xi)^2 + (y - \eta)^2 \right] d\xi d\eta.$$  

(5.121)

A direct evaluation of the double integral is not so easy. However, we can write down the potential in closed form by exploiting the following observations: First, thanks to (5.100, 119), the gravitational potential satisfies the Poisson equation

$$-\Delta u = \begin{cases} 
1, & \|x\| < 1, \\
0, & \|x\| > 1. 
\end{cases}$$  

(5.122)

Moreover, $u$ is clearly radially symmetric, and hence a function of $r$ alone. Thus, noting that $-\frac{1}{4} r^2 = -\frac{1}{4} x^2 - \frac{1}{4} y^2$ is a particular solution to $-\Delta u = 1$, we deduce that the potential (5.121) must be of the form

$$u(r) = \begin{cases} 
a + b \log r - \frac{1}{4} r^2, & r < 1, \\
c + d \log r, & r > 1, 
\end{cases}$$

where $a, b, c, d$ are constants. Now, for a non-concentrated mass, there cannot be a singularity at the origin, and so $b = 0$. Direct evaluation of (5.121) at $x = y = 0$, using polar coordinates, proves that $a = \frac{1}{4}$. Continuity of $u$ at $r = 1$ requires $c = a - \frac{1}{4} = 0$. Finally, let $C_R = \{ \|x\| = R \}$ be a circle of radius $R > 1$; combining (5.93) and (5.122), we have

$$-\pi = \iint_{\mathbb{R}^2} \Delta u \, dx \, dy = \oint_{\partial C_R} \frac{\partial u}{\partial n} \, ds = d \oint_{\partial C_R} \frac{\partial \log r}{\partial r} \, ds = 2\pi d,$$ 

and so $d = -\frac{1}{2}$.
Thus, the gravitational potential (5.121) due to a uniform disk of total mass (area) $\pi$ is, explicitly,

$$u(x, y) = \begin{cases} \frac{1}{4} (1 - r^2) = \frac{1}{4} (1 - x^2 - y^2), & x^2 + y^2 \leq 1, \\ -\frac{1}{2} \log r = -\frac{1}{2} \log (x^2 + y^2), & x^2 + y^2 \geq 1. \end{cases} \quad (5.123)$$

Observe that, outside the disk, the potential is exactly the same as the logarithmic potential due to a point mass of size $\pi$ located at the origin. In other words, the gravitational potential outside a uniform disk looks as if all its mass were concentrated at the origin.

With the free space logarithmic potential in hand, let us return to the question of finding the Green’s function for a boundary value problem on a bounded domain $\Omega \subset \mathbb{R}^2$. Since the logarithmic potential (5.108) is a particular solution to the Poisson equation (5.102), the general solution, according to Theorem 1.6, is given by $u = G_0 + z$ where $z$ is an arbitrary solution to the homogeneous equation $\Delta z = 0$, i.e., an arbitrary harmonic function. Thus, constructing the Green’s function has been reduced to the problem of finding the harmonic function $z(x, y)$ so that $G = G_0 + z$ satisfies the desired homogeneous boundary conditions. Let us explicitly formulate this result for the Dirichlet problem.

**Theorem 5.19.** The Green’s function for the Dirichlet boundary value problem for the Poisson equation on a bounded domain $\Omega \subset \mathbb{R}^2$ has the form

$$G(x, y; \xi, \eta) = G_0(x, y; \xi, \eta) + z(x, y; \xi, \eta), \quad (5.124)$$

where the first term is the logarithmic potential (5.108), while, for each $\xi = (\xi, \eta) \in \Omega$, the second term is the harmonic function that solves the boundary value problem

$$\Delta z = 0 \quad \text{on} \quad \Omega,$$

$$z(x, y; \xi, \eta) = \frac{1}{4\pi} \log \left[ (x - \xi)^2 + (y - \eta)^2 \right], \quad \text{for} \quad (x, y) \in \partial \Omega. \quad (5.125)$$

If $u(x, y)$ is a solution to the inhomogeneous Dirichlet problem

$$-\Delta u = f, \quad x \in \Omega, \quad u = h, \quad x \in \partial \Omega, \quad (5.126)$$

then

$$u(x, y) = \int \int_{\Omega} G(x, y; \xi, \eta) f(\xi, \eta) \, d\xi \, d\eta - \int_{\partial \Omega} \frac{\partial G}{\partial n}(x, y; \xi, \eta) h(\xi, \eta) \, ds. \quad (5.127)$$

**Proof:** To show that (5.124) is the Green’s function, we note that

$$-\Delta G = -\Delta G_0 - \Delta z = \delta_\xi \quad \text{in} \quad \Omega, \quad (5.128)$$

while

$$G(x, y; \xi, \eta) = G_0(x, y; \xi, \eta) + z(x, y; \xi, \eta) = 0 \quad \text{on} \quad \partial \Omega. \quad (5.129)$$

Next, to establish the solution formula (5.127), since both $z$ and $u$ are $C^2$, we can use (5.92) (with $v = z$, keeping in mind that $\Delta z = 0$) to establish

$$0 = -\int_{\Omega} z(x, y; \xi, \eta) \Delta u(\xi, \eta) \, d\xi \, d\eta$$

$$+ \int_{\partial \Omega} \left( z(x, y; \xi, \eta) \frac{\partial u}{\partial n}(\xi, \eta) - \frac{\partial z}{\partial n}(x, y; \xi, \eta) u(\xi, \eta) \right) \, ds.$$
Adding this to (5.109) and keeping (5.129) in mind, we deduce that

\[
    u(x, y) = -\int_{\Omega} \int G(x, y; \xi, \eta) \Delta u(\xi, \eta) \, d\xi \, d\eta - \int_{\partial \Omega} \frac{\partial G(x, y; \xi, \eta)}{\partial n} u(\xi, \eta) \, ds,
\]

which, given (5.126), produces (5.127).

Q.E.D.

The one subtle issue left unresolved is the existence of the solution. Read properly, Theorem 5.19 states that, if a classical solution exists, then it is necessarily given by the Green's function formula (5.127). Proving existence of the solution — and also the existence of the Green's function, or equivalently, the solution \( z \) to (5.125) — requires further in depth analysis, lying beyond the scope of these notes. In particular, unlike one-dimensional boundary value problems, mere continuity of the forcing function \( f \) is not quite sufficient to ensure the existence of a classical solution to the Poisson boundary value problem; differentiability is sufficient, although this assumption can be weakened. We refer to [73, 87], for a development of the Perron method based on approximating the solution by a sequence of subsolutions, which, by definition, solve the differential inequation \( -\Delta u \leq f \). An alternative proof using the direct method of the calculus of variations can be found in [36]. The latter proof relies on the characterization of the solution by a minimization principle, which we discuss in some detail in Chapter 9.

The Method of Images

The preceding analysis exposes the underlying form of the Green’s function, but we are still left with the determination of the harmonic component \( z(x, y) \) required to match the logarithmic potential boundary values, cf. (5.125). There are three principal analytical techniques employed to produce explicit formulas. The first is an adaptation of the method of separation of variables, and leads to infinite series expressions. We will not dwell on this approach here, although a couple of the exercises ask the reader to work through some of the details. The second is the method of images and will be developed in this section. The most powerful is based on the theory of conformal mappings, but must be deferred until we have learned the basics of complex analysis, as provided in Chapter 6. While the first two methods are limited to a fairly small class of domains, they do extend to higher dimensional problems, as well as certain other types of elliptic boundary value problems, whereas conformal mapping is, unfortunately, restricted to two-dimensional problems involving the Laplace and Poisson equations.

We already know that the singular part of the Green’s function for the two-dimensional Poisson equation is provided by a logarithmic potential. The problem, then, is to construct the harmonic part, called \( z(x, y) \) in (5.124), so that the sum has the correct homogeneous boundary values, or, equivalently, that \( z(x, y) \) has the same boundary values as the logarithmic potential. In certain cases, \( z(x, y) \) can be thought of as the potential induced by one or more hypothetical electric charges (or, equivalently, gravitational point masses) that are located outside the domain \( \Omega \), arranged in such a manner that their combined electrostatic potential happens to coincide with the logarithmic potential on the boundary of the domain. The goal, then, is to place image charges of suitable strength in the appropriate positions.
Here, we will only consider the case of a single image charge, located at a position \( \eta \not\in \Omega \). We scale the logarithmic potential (5.108) by the charge strength, and, for added flexibility, include an additional constant — the charge’s potential baseline:

\[
z(x, y) = a \log \|x - \eta\| + b, \quad \eta \in \mathbb{R}^2 \setminus \overline{\Omega}.
\]

The function \( z(x, y) \) is harmonic inside \( \Omega \) since the logarithmic potential is harmonic everywhere except at the external singularity \( \eta \). For the Dirichlet boundary value problem, then, for each point \( \xi \in \Omega \), we must find a corresponding image point \( \eta \in \mathbb{R}^2 \setminus \overline{\Omega} \) and constants \( a, b \in \mathbb{R} \), such that

\[
\log \|x - \xi\| = a \log \|x - \eta\| + b \quad \text{for all} \quad x \in \partial \Omega,
\]

or, equivalently,

\[
\|x - \xi\| = \lambda \|x - \eta\|^{a} \quad \text{for all} \quad x \in \partial \Omega,
\]

(5.130)

where \( \lambda = \log b \). For each fixed \( \xi, \eta, \lambda, a \), the equation in (5.130) will, typically, implicitly prescribe a plane curve, but it is not clear that one can always arrange that these curves all coincide with the boundary of our domain.

In order to make further progress, we appeal to a geometrical construction based upon similar triangles. Let us select \( \eta = c \xi \) to be a point lying on the ray through \( \xi \). Its location is chosen so that the triangle with vertices \( 0, x, \eta \) is similar to the triangle with vertices \( 0, \xi, x \), noting that they have the same angle at the common vertex \( 0 \) — see Figure 5.13. Similarity requires that the triangles’ corresponding sides have a common ratio, and so

\[
\frac{\|\xi\|}{\|x\|} = \frac{\|x - \xi\|}{\|\eta\|} = \frac{\|x - \eta\|}{\|x - \eta\|} = \lambda.
\]

(5.131)

The last equality implies that (5.130) holds with \( a = 1 \). Consequently, if we choose

\[
\|\eta\| = \frac{1}{\|\xi\|}, \quad \text{so that} \quad \eta = \frac{\xi}{\|\xi\|^2},
\]

(5.132)

\[\dagger\] To simplify the formulas, we have omitted the \( 1/(2\pi) \) factor, which can easily be reinstated at the end of the analysis.
then

\[ \| x \|^2 = \| \xi \| \| \eta \| = 1. \]

Thus \( x \) lies on the unit circle, and, as a result, \( \lambda = \| \xi \| \). The map taking a point \( \xi \) inside the disk to its image point \( \eta \) defined by (5.132) is known as \textit{inversion} with respect to the unit circle.

We have now demonstrated that the functions

\[ \frac{1}{2\pi} \log \| x - \xi \| = \frac{1}{2\pi} \log (\| \xi \| \| x - \eta \|) = \frac{1}{2\pi} \log \| \xi \|^2 \frac{\| x - \xi \|}{\| \xi \|} \quad \text{when} \quad \| x \| = 1, \tag{5.133} \]

has the same boundary values on the unit circle. Consequently, their difference

\[ G(x; \xi) = -\frac{1}{2\pi} \log \| x - \xi \| + \frac{1}{2\pi} \log \| \xi \|^2 \frac{\| x - \xi \|}{\| \xi \|} = \frac{1}{2\pi} \log \| \xi \|^2 \frac{\| x - \xi \|}{\| \xi \| \| x - \xi \|} \tag{5.134} \]

has the required properties for the Green’s function for the Dirichlet problem on the unit disk. In terms of polar coordinates

\[ x = (r \cos \theta, r \sin \theta), \quad \xi = (\rho \cos \phi, \rho \sin \phi), \]

applying the Law of Cosines to the triangles in Figure 5.13 leads to the explicit formula

\[ G(r, \theta; \rho, \phi) = \frac{1}{4\pi} \log \left( \frac{1 + r^2 \rho^2 - 2r \rho \cos(\theta - \phi)}{r^2 + \rho^2 - 2r \rho \cos(\theta - \phi)} \right). \tag{5.135} \]

In Figure 5.14 we sketch the Green’s function for the Dirichlet boundary value problem corresponding to a unit impulse being applied at a point half way between the center and
the edge of the disk. We also require its normal, or radial, derivative on a circle:

$$\frac{\partial G}{\partial r} (r, \theta; \rho, \phi) = - \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)}. \tag{5.136}$$

Thus, specializing (5.127), we arrive at a solution to the Dirichlet boundary value problem for the Poisson equation in the unit disk.

**Theorem 5.20.** The solution to the inhomogeneous Dirichlet boundary value problem

$$-\Delta u = f, \quad \text{for} \quad r = \|x\| < 1, \quad u = h, \quad \text{for} \quad r = 1,$$

is, when expressed in polar coordinates,

$$u(r, \theta) = \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{1} f(\rho, \phi) \log \left( \frac{1 + r^2 \rho^2 - 2r \rho \cos(\theta - \phi)}{r^2 + \rho^2 - 2r \rho \cos(\theta - \phi)} \right) \rho \, d\rho \, d\phi + \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)} \, d\phi. \tag{5.137}$$

In particular, when $f \equiv 0$, (5.137) recovers the Poisson integral formula (4.117) for the solution to the Dirichlet boundary value problem for the Laplace equation.

**Example 5.21.** A particularly important case is when $f \equiv 0$ and the boundary value

$$h(\theta) = \delta(\theta - \phi)$$
is a delta function concentrated at the point \((\cos \phi, \sin \phi), -\pi < \phi \leq \pi\), on the unit circle. The solution to the resulting boundary value problem is the **Poisson integral kernel**

\[
  u(r, \theta) = \frac{1 - r^2}{2\pi[1 + r^2 - 2r \cos(\theta - \phi)]},
\]  

(5.138)

that we first constructed when establishing (4.117); the solution is sketched in Figure 5.15. The reader may enjoy verifying that this function does indeed solve the Laplace equation and has the correct boundary values in the limit as \(r \to 1\). Physically, if \(u(r, \theta)\) represents the equilibrium temperature of the disk, then the delta function boundary data correspond to a concentrated unit heat source applied to a single point on the boundary. Thus, the Poisson kernel plays the role of the fundamental solution for the boundary value problem. Indeed, Poisson integral formula (4.117) follows from our general superposition principle, in which we write the boundary data as a superposition of delta functions:

\[
  h(\theta) = \int_{-\pi}^{\pi} h(\phi) \delta(\phi - \theta) d\phi,
\]